

Citation for published version:

Moiola, A & Spence, E 2019, 'Acoustic transmission problems: wavenumber-explicit bounds and resonance-free regions', *Mathematical Models & Methods in Applied Sciences*, vol. 29, no. 2, pp. 317-354.
<https://doi.org/10.1142/S0218202519500106>

DOI:

[10.1142/S0218202519500106](https://doi.org/10.1142/S0218202519500106)

Publication date:

2019

Document Version

Peer reviewed version

[Link to publication](#)

Electronic version of an article published as *Mathematical Models and Methods in Applied Sciences*, Volume 29, Issue 2, 2019, Pages 317-354, <https://doi.org/10.1142/S0218202519500106>. © Copyright World Scientific Publishing Company, 2019. <https://www.worldscientific.com/worldscinet/m3as>

University of Bath

Alternative formats

If you require this document in an alternative format, please contact:
openaccess@bath.ac.uk

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Acoustic transmission problems: wavenumber-explicit bounds and resonance-free regions

Andrea Moiola*, Euan A. Spence†

28th November 2018

Abstract

We consider the Helmholtz transmission problem with one penetrable star-shaped Lipschitz obstacle. Under a natural assumption about the ratio of the wavenumbers, we prove bounds on the solution in terms of the data, with these bounds explicit in all parameters. In particular, the (weighted) H^1 norm of the solution is bounded by the L^2 norm of the source term, independently of the wavenumber. These bounds then imply the existence of a resonance-free strip beneath the real axis. The main novelty is that the only comparable results currently in the literature are for smooth, convex obstacles with strictly positive curvature, while here we assume only Lipschitz regularity and star-shapedness with respect to a point. Furthermore, our bounds are obtained using identities first introduced by Morawetz (essentially integration by parts), whereas the existing bounds use the much-more sophisticated technology of microlocal analysis and propagation of singularities. We also adapt existing results to show that if the assumption on the wavenumbers is lifted, then no bound with polynomial dependence on the wavenumber is possible.

AMS subject classification: 35B34, 35J05, 35J25, 78A45

Keywords: transmission problem, resonance, Helmholtz equation, acoustic, frequency explicit, wavenumber explicit, Lipschitz domain, Morawetz identity, semiclassical

1 Introduction

The acoustic transmission problem, modelled by the Helmholtz equation, is a classic problem in scattering theory. Despite having been studied from many different perspectives over the years, it remains a topic of active research. For example, recent research on this problem includes the following.

- Designing novel integral-equation formulations of the transmission problem; see, e.g., [7] and the reviews [18, 19].
- Designing hybrid numerical-asymptotic methods to approximate the solution of the transmission problem with a number of degrees of freedom that grows slowly (or, ideally, is constant) as the wavenumber increases [32, 33].
- Quantifying how uncertainty in the shape of the obstacle affects the solution of the transmission problem [37].
- Obtaining sharp bounds on the location of resonances of the transmission problem [28].
- Obtaining sharp estimates on the scattered field away from the obstacle, for example in the case when the obstacle is a ball [8, 9, 35, 53], motivated by applications in imaging, inverse problems, and cloaking.
- Designing fast solvers for the Helmholtz equation in media where the wavenumber is piecewise smooth (i.e. transmission problems); see, e.g., the recent review [29] and the references therein.
- Proving wavenumber-explicit bounds on solutions of boundary value problems that approximate transmission problems, motivated by applications in numerical analysis [3, 17, 31, 59].

*Department of Mathematics, University of Pavia, 27100 Pavia, Italy (andrea.moiola@unipv.it)

†Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY (E.A.Spence@bath.ac.uk)

In this paper we focus on the case of transmission through one obstacle; i.e. the problem has two real wavenumbers: one inside the obstacle, and one outside the obstacle. We give a precise definition of this problem in Equation (2.2) below. Our results can also be extended to more general situations (see Remark 3.4 below).

A natural question to ask about the transmission problem is:

- Q1. Can one find a bound on the solution in terms of the data, with the bound explicit in the two wavenumbers?

Ideally, we would also like the bound to be either independent of the shape of the scatterer, or explicit in any of its natural geometric parameters; for example, if the domain is star-shaped, then we would ideally like the bound to be explicit in the star-shapedness parameter. Another fundamental question is

- Q2. Does the solution operator (thought of as a function of the wavenumber) have a resonance-free region underneath the real axis?

The relationship between Q1, Q2, and the question of local-energy decay for solutions of the corresponding wave equation is well-understood in scattering theory, and goes back to the work of Lax, Morawetz, and Phillips. In this particular situation of the Helmholtz transmission problem, Vodev proved in [69, Theorem 1.1 and Lemma 2.3] that an appropriate bound on the solution for real wavenumbers implies the existence of a resonance-free strip beneath the real axis.

Existing work on Q1 and Q2 for the transmission problem. To the authors' knowledge, there are five main sets of results regarding Q1 and Q2 for the Helmholtz transmission problem in the literature; we highlight that several of these results cover more general transmission problems than the single-penetrable-obstacle one considered in this paper.

- (a) When the wavenumber outside the obstacle is *larger* than the wavenumber inside the obstacle, and the obstacle is C^∞ and convex with strictly positive curvature, Cardoso, Popov, and Vodev proved that the solution can be bounded independent of the wavenumber, and thus that there exists a resonance-free strip beneath the real axis [10] (these results were an improvement of the earlier work by Popov and Vodev [56]).
- (b) When the wavenumber outside the obstacle is *smaller* than the wavenumber inside the obstacle, and the obstacle is C^∞ and convex with strictly positive curvature, Popov and Vodev proved that there exists a sequence of complex wavenumbers (lying super-algebraically close to the real axis) through which the norm of the solution grows faster than any algebraic power of the wavenumber [57].
- (c) For either configuration of wavenumbers, and for any C^∞ obstacle, Bellassoued proved that the norm of the solution cannot grow faster than exponentially with the wavenumber [5].
- (d) Further information about the location and the asymptotics of the resonances when the obstacle is C^∞ and convex with strictly positive curvature, for both wavenumber configurations above, was obtained by Cardoso, Popov, and Vodev in [11]. Sharp bounds on the location of the resonances (again for both configurations of the wavenumbers) were given recently by Galkowski in [28].
- (e) The case when the obstacle is a ball, in both configurations of the wavenumbers, has been studied by Capdeboscq and coauthors in [8, 9] (and summarised in [1, Chapter 5]) using separation of variables and bounds on Bessel and Hankel functions.

The main results of this paper and their novelty. In this paper we prove analogues of the bound in (a) above when the obstacle is Lipschitz and star-shaped (Theorems 3.1 and 3.2) and hence also the existence of a resonance-free strip beneath the real axis (Theorem 3.5). Our condition on the ratio of the wavenumbers is slightly more restrictive than that in [10], since the parameters in the transmission conditions are also involved (see Equation (3.1) and Remark 5.1 below). Nevertheless we believe this is the first time such results have been proved for the transmission problem when the obstacle is non-convex or non-smooth.

Furthermore, the constant in our bound is given completely explicitly, and the bound is valid for all wavenumbers greater than zero (satisfying the restriction on the ratio). On the other hand,

the bound for smooth convex obstacles in [10] assumes that both wavenumbers are large, with the ratio of the two fixed, but the bound is not explicit in this ratio (although we expect the results of [28] could be used to get a bound for smooth convex obstacles that is explicit in the ratio of wavenumbers, when the wavenumbers are large enough [27]). An additional feature of the constant in our bound is that, whilst the bound is valid for all star-shaped Lipschitz obstacles, the constant only depends on the diameter of the obstacle. This “shape-robustness” makes the bound particular suitable for applications in quantifying how uncertainty in the shape of the obstacle affects the solution (done for the transmission problem at low frequency in [37]), and these applications are currently under investigation.

We highlight that the bound in [10] relies on microlocal analysis and the deep results of Melrose and Sjöstrand on propagation of singularities [45, 46]. In contrast, our bound is obtained using identities for solutions of the Helmholtz equation first introduced by Morawetz in [50, 51], which boil down to multiplying the PDEs by carefully-chosen test functions and integrating by parts. Whereas Morawetz’s identities have been used to prove bounds on many Helmholtz BVPs, famously the exterior Dirichlet and Neumann problems in [50, 51], it appears that (surprisingly) this paper is their first application to the Helmholtz transmission problem involving one penetrable obstacle (Remark 3.8 below discusses their applications to transmission problems not involving a bounded obstacle). The novelty of this paper is therefore not in the techniques that are used, but the fact that these well-known techniques can be applied to a classic problem to obtain new results.

Outline of the paper. In §2 we define the Helmholtz transmission problem for Lipschitz obstacles and recap results on existence, uniqueness, and regularity. In §3 we give the main results (Theorems 3.1, 3.2 and 3.5). In §4 we derive the Morawetz identities used in the proofs of Theorems 3.1 and 3.2, and in §5 we prove the main results. In §6 we adapt the existing results of [57] about super-algebraic growth of the norm of the solution through a sequence of complex wavenumbers to prove analogous growth through a sequence of real wavenumbers; we illustrate this growth through real wavenumbers with plots when the obstacle is a 2-d ball.

Motivation for §6. Our motivation for adapting the results of [57] (and also highlighting the results of [8, 9]) in §6 is the recent investigations [3, 17, 31, 59] of the interior impedance problem for the Helmholtz equation with piecewise-constant wavenumber (and the related investigation [54] for piecewise-Lipschitz wavenumber). These investigations, coming from the numerical-analysis community, concern Helmholtz transmission problems, but consider the interior impedance problem, because the impedance boundary condition is a simple approximation of the Sommerfeld radiation condition that is commonly used when implementing finite-element methods (see Remark 3.7 below).

The results of [57], adapted as in §6 (using, in particular, recent wavenumber-explicit bounds on layer-potential operators from [34, 61]),

- (a) show that the “technical” assumptions on the wavenumber in [3, §1] are in fact necessary for the results of [3, §1] to hold, and
- (b) partially answer the conjecture in [59, §2.3] about the maximal growth of the norm of the solution operator.

2 Formulation of the problem

2.1 Geometric notation.

Let $\Omega_i \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz open set. Denote $\Omega_o := \mathbb{R}^d \setminus \overline{\Omega_i}$ and $\Gamma := \partial\Omega_i = \partial\Omega_o$. Let \mathbf{n} be the unit normal vector field on Γ pointing from Ω_i into Ω_o . We denote by $\partial_{\mathbf{n}}$ the corresponding Neumann trace from one of the two domains Ω_i and Ω_o and we do not use any symbol for the Dirichlet trace on Γ . For any $\varphi \in L^2_{\text{loc}}(\mathbb{R}^d)$, we write $\varphi_i := \varphi|_{\Omega_i}$ and $\varphi_o := \varphi|_{\Omega_o}$.

For $a > 0$ and $\mathbf{x}_0 \in \mathbb{R}^d$ we denote by $B_a(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x} - \mathbf{x}_0| < a\}$ the ball with centre \mathbf{x}_0 and radius a ; if $\mathbf{x}_0 = \mathbf{0}$ we write $B_a = B_a(\mathbf{0})$. Given $R > 0$ such that $\overline{\Omega_i} \subset B_R$, let $D_R := B_R \cap \Omega_o$ and $\Gamma_R := \partial B_R = \{|\mathbf{x}| = R\}$. On Γ_R the unit normal \mathbf{n} points outwards. With D denoting an open set or a $d - 1$ -dimensional manifold, $\|\cdot\|_D$ denotes $L^2(D)$ norm for scalar or vector fields. On Γ and Γ_R , ∇_T denotes the tangential gradient.

To state the main results, we need to define the notions of *star-shaped* and *star-shaped with respect to a ball*.

Definition 2.1. (i) Ω_i is star-shaped with respect to the point \mathbf{x}_0 if, whenever $\mathbf{x} \in \Omega_i$, the segment $[\mathbf{x}_0, \mathbf{x}] \subset \Omega_i$.
(ii) Ω_i is star-shaped with respect to the ball $B_a(\mathbf{x}_0)$ if it is star-shaped with respect to every point in $B_a(\mathbf{x}_0)$.

These definitions make sense even for non-Lipschitz Ω_i , but when Ω_i is Lipschitz one can characterise star-shapedness with respect to a point or ball in terms of $(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x})$ for $\mathbf{x} \in \Gamma$.

Lemma 2.2. (i) If Ω_i is Lipschitz, then it is star-shaped with respect to \mathbf{x}_0 if and only if $(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \Gamma$ for which $\mathbf{n}(\mathbf{x})$ is defined.
(ii) Ω_i is star-shaped with respect to $B_a(\mathbf{x}_0)$ if and only if it is Lipschitz and $(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x}) \geq a$ for all $\mathbf{x} \in \Gamma$ for which $\mathbf{n}(\mathbf{x})$ is defined;

Proof. See [47, Lemma 5.4.1] or [36, Lemma 3.1]. \square

In the rest of the paper, whenever Ω_i is star-shaped with respect to a point or ball, we assume (without loss of generality) that $\mathbf{x}_0 = \mathbf{0}$.

2.2 The Helmholtz transmission problem.

From the point of view of obtaining wavenumber-explicit bounds, we are interested in the case when the wavenumber is real. Nevertheless, in order to talk about resonance-free regions, we must also consider complex wavenumbers.

Definition 2.3. (Sommerfeld radiation condition.) Given $\varphi \in C^1(\mathbb{R}^d \setminus B_R)$, for some ball $B_R = \{|\mathbf{x}| < R\}$, and $\kappa \in \mathbb{C} \setminus \{0\}$ with $\Im \kappa \geq 0$, we say that φ satisfies the Sommerfeld radiation condition if

$$\lim_{r \rightarrow \infty} r^{\frac{d-1}{2}} \left(\frac{\partial \varphi(\mathbf{x})}{\partial r} - i\kappa \varphi(\mathbf{x}) \right) = 0 \quad (2.1)$$

uniformly in all directions, where $r = |\mathbf{x}|$; we then write $\varphi \in \text{SRC}(\kappa)$.

Recall that when $\Im \kappa > 0$, if φ satisfies the Sommerfeld radiation condition, then φ decays exponentially at infinity (see, e.g., [21, Theorem 3.6]).

Definition 2.4. (The Helmholtz transmission problem.) Let $k \in \mathbb{C} \setminus \{0\}$ with $\Im k \geq 0$, and let $n_i, n_o, a_i, a_o, A_D, A_N$ be positive real numbers. Let $f_i \in L^2(\Omega_i)$, $f_o \in L^2(\Omega_o)$, $g_D \in H^1(\Gamma)$, $g_N \in L^2(\Gamma)$, and assume f_o has compact support. The Helmholtz transmission problem is: find $u \in H_{\text{loc}}^1(\mathbb{R}^d \setminus \Gamma)$ such that,

$$\begin{aligned} a_i \Delta u_i + k^2 n_i u_i &= f_i && \text{in } \Omega_i, \\ a_o \Delta u_o + k^2 n_o u_o &= f_o && \text{in } \Omega_o, \\ u_o &= A_D u_i + g_D && \text{on } \Gamma, \\ a_o \partial_{\mathbf{n}} u_o &= A_N a_i \partial_{\mathbf{n}} u_i + g_N && \text{on } \Gamma, \\ u_o &\in \text{SRC}(k\sqrt{n_o/a_o}). \end{aligned} \quad (2.2)$$

Four of the parameters $n_i, n_o, a_i, a_o, A_D, A_N$ are redundant; in particular we can set either $A_D = a_i = a_o = n_o = 1$ or $A_D = A_N = a_o = n_o = 1$ and still cover all problems by rescaling the remaining coefficients, u_i , and the source terms. Nevertheless, we keep all six parameters in (2.2) since given a specific problem it is then easy to write it in the form (2.2), setting some parameters to one.

Some extensions to transmission problems more general than those in Definition 2.4 are discussed in Remark 3.4.

Remark 2.5. (Relation to acoustics and electromagnetics.) Time-harmonic acoustic transmission problems are often written in the form

$$\operatorname{div} \left(\frac{1}{\rho} \nabla u \right) + \frac{\kappa^2}{\rho} u = F, \quad u \in H^1(\mathbb{R}^d), \quad \frac{1}{\rho} \nabla u \in H(\operatorname{div}; \mathbb{R}^d), \quad (2.3)$$

and u satisfies the Sommerfeld radiation condition, where $\rho(\mathbf{x})$ and $\kappa(\mathbf{x})$ are positive functions; see e.g. [39, eq. (1)]. (Recall that $\mathbf{v} \in H(\operatorname{div}; \mathbb{R}^d)$ if and only if $\mathbf{v}|_{\Omega_i} \in H(\operatorname{div}; \Omega_i)$, $\mathbf{v}|_{\Omega_o} \in H(\operatorname{div}; \Omega_o)$)

and $\mathbf{v}|_{\Omega_i} \cdot \mathbf{n} = \mathbf{v}|_{\Omega_o} \cdot \mathbf{n}$ in $H^{-1/2}(\Gamma)$.) In the particular case where ρ and κ take two different values on Ω_i and Ω_o , problem (2.3) can be written in the form (2.2) choosing, for example,

$$A_D = A_N = 1, \quad a = \frac{1}{\rho}, \quad k = \kappa_o, \quad n_o = \frac{1}{\rho_o}, \quad n_i = \left(\frac{\kappa_i}{\kappa_o}\right)^2 \frac{1}{\rho_i}, \quad f = F, \quad g_D = g_N = 0,$$

or

$$A_D = a_i = a_o = n_o = 1, \quad A_N = \frac{\rho_o}{\rho_i}, \quad k = \kappa_o, \quad n_i = \left(\frac{\kappa_i}{\kappa_o}\right)^2, \quad f = \rho F, \quad g_D = g_N = 0.$$

(More generally, one can choose any constant $k > 0$ and then let $n = a\kappa^2/k^2$.)

The time-harmonic Maxwell equations are

$$\operatorname{curl} \mathbf{H} + ik\varepsilon \mathbf{E} = (ik)^{-1} \mathbf{J}, \quad \operatorname{curl} \mathbf{E} - ik\mu \mathbf{H} = \mathbf{0} \quad \text{in } \mathbb{R}^3. \quad (2.4)$$

When all fields and parameters involved depend only on two Cartesian space variables, say x and y , Equations (2.4) reduce to the (heterogeneous) Helmholtz equation in \mathbb{R}^2 . In the transverse-magnetic (TM) mode, \mathbf{J} and \mathbf{E} are given by $\mathbf{J} = (0, 0, J_z(x, y))$ and $\mathbf{E} = (0, 0, E_z(x, y))$, so (2.4) reduce to a scalar equation for the third component of the electric field:

$$\operatorname{div} \left(\frac{1}{\mu} \nabla E_z \right) + k^2 \varepsilon E_z = -J_z.$$

If the permittivity ε and the permeability μ are constant in $\widetilde{\Omega}_i = \Omega_i \times \mathbb{R} \subset \mathbb{R}^3$ and $\widetilde{\Omega}_o = \Omega_o \times \mathbb{R} \subset \mathbb{R}^3$ for $\Omega_i, \Omega_o \subset \mathbb{R}^2$ as in §2.1, then (2.4) (supplemented with suitable radiation conditions) can be written as Problem (2.2) for $u = E_z$ in \mathbb{R}^2 with $a = 1/\mu$, $n = \varepsilon$, $A_D = A_N = 1$. Vice versa, in the transverse-electric (TE) mode, $\mathbf{J} = (J_x(x, y), J_y(x, y), 0)$, $\mathbf{H} = (0, 0, H_z(x, y))$ and

$$\operatorname{div} \left(\frac{1}{\varepsilon} \nabla H_z \right) + k^2 \mu H_z = (ik\varepsilon)^{-1} \left(\frac{\partial J_x}{\partial y} - \frac{\partial J_y}{\partial x} \right),$$

so (2.4) can be written as (2.2) for $u = H_z$ with $a = 1/\varepsilon$, $n = \mu$, $A_D = A_N = 1$. Observe that for TM and TE modes, the parameters n_i, n_o, a_i, a_o depend on properties of the medium through which the waves propagate, whereas k depends on the wave itself.

Definition 2.6. (Scattering problem.) Let $k \in \mathbb{C} \setminus \{0\}$ with $\Im k \geq 0$, and let $n_i, n_o, a_i, a_o, A_D, A_N$ be positive real numbers. Let u^I be a solution of $a_o \Delta u^I + k^2 n_o u^I = 0$ that is C^∞ in a neighbourhood of $\overline{\Omega}_i$ (for example a plane wave, a circular or spherical wave, or a fundamental solution centred in Ω_o). Define the total field u^T to be solution of

$$\begin{aligned} a_i \Delta u_i^T + k^2 n_i u_i^T &= 0 && \text{in } \Omega_i, \\ a_o \Delta u_o^T + k^2 n_o u_o^T &= 0 && \text{in } \Omega_o, \\ u_o^T &= A_D u_i^T && \text{on } \Gamma, \\ a_o \partial_{\mathbf{n}} u_o^T &= A_N a_i \partial_{\mathbf{n}} u_i^T && \text{on } \Gamma, \\ (u_o^T - u^I) &\in \operatorname{SRC}(k\sqrt{n_o/a_o}). \end{aligned} \quad (2.5)$$

The scattered field defined by $u := u^T - u^I$ satisfies

$$\begin{aligned} a_i \Delta u_i + k^2 n_i u_i &= k^2 \left(\frac{a_i}{a_o} n_o - n_i \right) u^I =: f_i && \text{in } \Omega_i, \\ a_o \Delta u_o + k^2 n_o u_o &= 0 && \text{in } \Omega_o, \\ u_o &= A_D u_i + (A_D - 1) u^I && \text{on } \Gamma, \\ a_o \partial_{\mathbf{n}} u_o &= A_N a_i \partial_{\mathbf{n}} u_i + (A_N - 1) a_i \partial_{\mathbf{n}} u^I && \text{on } \Gamma, \\ u_o &\in \operatorname{SRC}(k\sqrt{n_o/a_o}). \end{aligned} \quad (2.6)$$

The scattering problem (2.6) can therefore be written in the form (2.2) for $f_i = k^2 \left(\frac{a_i}{a_o} n_o - n_i \right) u^I$, $f_o = 0$, $g_D = (A_D - 1) u^I$ and $g_N = (A_N - 1) a_i \partial_{\mathbf{n}} u^I$.

The next lemma, proved in Appendix A, addresses the questions of existence, uniqueness, and regularity of the solution of (2.2).

Lemma 2.7. (Existence, uniqueness, and regularity.) *The Helmholtz transmission problem of Definition 2.4 admits a unique solution $u \in H_{\text{loc}}^1(\mathbb{R}^d \setminus \Gamma)$. Moreover $u_i, u_o \in H^1(\Gamma)$ and $\partial_{\mathbf{n}} u_i, \partial_{\mathbf{n}} u_o \in L^2(\Gamma)$.*

The key point about Lemma 2.7 is that, whilst the existence and uniqueness results are well known, the regularity results $u_i, u_o \in H^1(\Gamma)$ and $\partial_{\mathbf{n}} u_i, \partial_{\mathbf{n}} u_o \in L^2(\Gamma)$ are currently only available in the literature in the case $d = 3$, $f_i = f_o = 0$, and $k \in \mathbb{R}$. These results are consequences of the harmonic-analysis results about layer potentials in [20, 25, 68] and the regularity results of Nečas for strongly elliptic systems in [52, §5.1.2 and §5.2.1], [44, Theorem 4.24], and are needed to apply the Morawetz identities of §4 to the solution of the transmission problem when Ω_i is Lipschitz.

Finally, recall that k is a resonance of the boundary value problem (2.2) if there exists a non-zero u_o satisfying (2.2) with $f_i = f_o = g_D = g_N = 0$ and the Sommerfeld radiation condition replaced by

$$u_o(\mathbf{x}) = \frac{e^{ik\sqrt{n_o/a_o}r}}{r^{(d-1)/2}} \left(u_\infty(\hat{\mathbf{x}}) + \mathcal{O}\left(\frac{1}{r}\right) \right) \quad \text{as } r := |\mathbf{x}| \rightarrow \infty, \quad (2.7)$$

for some function u_∞ of $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ (the far-field pattern); see, e.g., [24, §3.6 and Theorem 4.9] or [40, §2]. The uniqueness result of Lemma 2.7 implies that any resonance must have $\Im k < 0$, and thus (2.7) implies that u_o grows exponentially at infinity.

3 Main results

3.1 Bounds on the solution (answering Q1)

In this section we assume that $k > 0$, but analogous results hold for $k < 0$ as well. We recall that the main assumptions we stipulate, namely (3.1) and (3.3) below, mean that the wavenumber is larger in the exterior region Ω_o than in the interior region Ω_i , or equivalently that the wavelength $\lambda = \frac{2\pi\sqrt{a}}{k\sqrt{n}}$ is longer in Ω_i than in Ω_o . We recall from §2.1 that the notation $\|\cdot\|_D$ stands for the $L^2(D)$ norm.

Theorem 3.1. *Assume that Ω_i is star-shaped,*

$$\frac{n_i}{n_o} \leq \frac{A_D}{A_N} \leq \frac{a_i}{a_o}, \quad (3.1)$$

$g_N = g_D = 0$ and $k > 0$. Given $R > 0$ such that $\text{supp } f_o \subset B_R$, recall that $D_R := \Omega_o \cap B_R$. The solution of BVP (2.2) then satisfies

$$\begin{aligned} a_i \|\nabla u_i\|_{\Omega_i}^2 + k^2 n_i \|u_i\|_{\Omega_i}^2 + \frac{1}{A_D A_N} \left(a_o \|\nabla u_o\|_{D_R}^2 + k^2 n_o \|u_o\|_{D_R}^2 \right) \\ \leq \left[\frac{4 \text{diam}(\Omega_i)^2}{a_i} + \frac{1}{n_i} \left(2\sqrt{\frac{n_o}{a_o}} R + \frac{d-1}{k} \right)^2 \right] \|f_i\|_{\Omega_i}^2 \\ + \frac{1}{A_D A_N} \left[\frac{4R^2}{a_o} + \frac{1}{n_o} \left(2\sqrt{\frac{n_o}{a_o}} R + \frac{d-1}{k} \right)^2 \right] \|f_o\|_{D_R}^2. \end{aligned} \quad (3.2)$$

The bound (3.2) is valid for all star-shaped Lipschitz Ω_i , but the constants on the right-hand side only depend on Ω_i via $\text{diam}(\Omega_i)$. As highlighted in §1, we expect that this uniformity of the bound with respect to the geometry makes it particular suitable for applications in quantifying how uncertainty in the shape of Ω_i affects the solution (as done for small k in [37]).

In Theorem 3.1 we assumed that the boundary source terms g_D and g_N vanish. In the next theorem we consider general $g_D \in H^1(\Gamma)$ and $g_N \in L^2(\Gamma)$. In order to do this, we need to assume that the inequalities (3.1) on the parameters are strict and that Ω_i is star-shaped with respect to a ball.

Theorem 3.2. *Assume that Ω_i is star-shaped with respect to $B_{\gamma \text{diam}(\Omega_i)}$ for some $0 < \gamma \leq 1/2$,*

$$\frac{n_i}{n_o} < \frac{A_D}{A_N} < \frac{a_i}{a_o}, \quad (3.3)$$

$k > 0$ and $R > 0$ is such that $\text{supp } f_o \subset B_R$. Then the solution of (2.2) satisfies

$$\begin{aligned}
& a_i \|\nabla u_i\|_{\Omega_i}^2 + k^2 n_i \|u_i\|_{\Omega_i}^2 + \frac{1}{A_D A_N} \left(a_o \|\nabla u_o\|_{D_R}^2 + k^2 n_o \|u_o\|_{D_R}^2 \right) \\
& \leq \left[\frac{4 \text{diam}(\Omega_i)^2}{a_i} + \frac{1}{n_i} \left(2\sqrt{\frac{n_o}{a_o}} R + \frac{d-1}{k} \right)^2 \right] \|f_i\|_{\Omega_i}^2 \\
& \quad + \frac{1}{A_D A_N} \left[\frac{4R^2}{a_o} + \frac{1}{n_o} \left(2\sqrt{\frac{n_o}{a_o}} R + \frac{d-1}{k} \right)^2 \right] \|f_o\|_{D_R}^2 \\
& \quad + 2 \left[\frac{\text{diam}(\Omega_i) a_o ((3+2\gamma) a_i A_N + 2a_o A_D)}{A_D A_N \gamma (a_i A_N - a_o A_D)} \right] \|\nabla_T g_D\|_{\Gamma}^2 \\
& \quad + 2 \left[\frac{2 \text{diam}(\Omega_i) n_o^2}{\gamma A_N (n_o A_D - n_i A_N)} + \frac{(3+\gamma) a_i \left(n_o R^2 + \frac{a_o (d-1)^2}{4k^2} \right)}{\gamma A_D \text{diam}(\Omega_i) (a_i A_N - a_o A_D)} \right] k^2 \|g_D\|_{\Gamma}^2 \\
& \quad + \frac{2}{\gamma a_o A_N A_D} \left[\frac{\text{diam}(\Omega_i) (4a_i A_N + 2a_o A_D)}{a_i A_N - a_o A_D} + \frac{2A_D \left(n_o R^2 + \frac{a_o (d-1)^2}{4k^2} \right)}{\text{diam}(\Omega_i) (n_o A_D - n_i A_N)} \right] \|g_N\|_{\Gamma}^2.
\end{aligned} \tag{3.4}$$

Note that each of the coefficients in front of the norms on the right-hand side of the bound (3.4) is a non-increasing function of k , apart from the coefficient k^2 multiplying $\|g_D\|_{\Gamma}^2$.

Corollary 3.3. (Theorems 3.1 and 3.2 applied to the scattering problem (2.6).) *The solution u of the scattering problem (2.6) with $n_o = a_o = A_D = A_N = 1$, $n_i \leq 1 \leq a_i$, and Ω_i star-shaped satisfies*

$$\begin{aligned}
& a_i \|\nabla u_i\|_{\Omega_i}^2 + k^2 n_i \|u_i\|_{\Omega_i}^2 + \|\nabla u_o\|_{D_R}^2 + k^2 \|u_o\|_{D_R}^2 \\
& \leq \left[\frac{4 \text{diam}(\Omega_i)^2}{a_i} + \frac{1}{n_i} \left(2R + \frac{d-1}{k} \right)^2 \right] k^4 (a_i - n_i)^2 \|u^I\|_{\Omega_i}^2.
\end{aligned}$$

The solution u of the scattering problem (2.6) with $n_o = a_o = a_i = A_D = 1$, $n_i < 1/A_N < 1$ and Ω_i star-shaped with respect to $B_{\gamma \text{diam}(\Omega_i)}$ (for some $0 < \gamma \leq 1/2$) satisfies

$$\begin{aligned}
& \|\nabla u_i\|_{\Omega_i}^2 + k^2 n_i \|u_i\|_{\Omega_i}^2 + \frac{1}{A_N} \left(\|\nabla u_o\|_{D_R}^2 + k^2 \|u_o\|_{D_R}^2 \right) \\
& \leq \left[4 \text{diam}(\Omega_i)^2 + \frac{1}{n_i} \left(2R + \frac{d-1}{k} \right)^2 \right] k^4 (1 - n_i)^2 \|u^I\|_{\Omega_i}^2 \\
& \quad + \frac{4}{\gamma A_N} \left[\frac{\text{diam}(\Omega_i) (2A_N + 1)}{A_N - 1} + \frac{R^2 + \frac{(d-1)^2}{4k^2}}{\text{diam}(\Omega_i) (1 - n_i A_N)} \right] (A_N - 1)^2 \|\partial_{\mathbf{n}} u^I\|_{\Gamma}^2.
\end{aligned}$$

Remark 3.4. (Extensions of Theorems 3.1 and 3.2.) *We have only considered the case of a single penetrable obstacle with a_i, a_o, n_i , and n_o all real and constant, but analogues of Theorems 3.1 and 3.2 hold in the following cases (and also when the cases are combined).*

1. When there are multiple “layers”, each with constant a and n , and with the boundaries of the layers star-shaped with respect to the origin (for the analogue of Theorem 3.1) or star-shaped with respect to balls centred at the origin (for the analogue of Theorem 3.2); in this case the conditions (3.1)/(3.3) must hold at each interface.
2. When a_i, a_o, n_i , and n_o are functions of position and satisfy conditions that ensure nontrapping of rays.
3. When Ω_i contains an impenetrable star-shaped Dirichlet scatterer.
4. When Ω_o is truncated by a star-shaped boundary and the radiation condition is approximated by an impedance boundary condition.
5. When n_i is complex with $0 < \Im n_i \leq \delta/k$, for δ a sufficiently small constant; this models a particular case of a lossy scatterer in a lossless background.
6. When Condition (3.1) is partly violated, namely when n_i/n_o is slightly larger (in a k -dependent way) than A_D/A_N and a_i/a_o .

The extension to Case 1 is clear from the proofs in §5, the extension to Case 2 and 3 are covered in [30], the extension to the Case 4 is discussed in Remark 4.5, the extension to Case 5 is discussed in Remarks 3.9 and 5.2, and the extension to Case 6 is described in Proposition 5.3.

3.2 Resonance-free strip (answering Q2)

We let $R(k)$ denote the solution operator of the Helmholtz transmission problem of Definition 2.4 when g_D and g_N are both zero, i.e.

$$R(k) : \begin{pmatrix} f_i \\ f_o \end{pmatrix} \mapsto \begin{pmatrix} u_i \\ u_o \end{pmatrix}.$$

Although $R(k)$ depends also on the parameters n_o, n_i, a_o, a_i, A_D , and A_N , in what follows we consider these fixed and consider k as variable. Let $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^d)$ such that $\chi_j \equiv 1$ in a neighbourhood of Ω_i , and let

$$R_\chi(k) := \chi_1 R(k) \chi_2; \quad (3.5)$$

i.e. $R_\chi(k)$ is the cut-off resolvent. Then

$$R_\chi(k) : L^2(\Omega_i) \oplus L^2(\Omega_o) \rightarrow H^1(\Omega_i) \oplus H^1(\Omega_o)$$

for $k \in \mathbb{R} \setminus \{0\}$.

Theorem 3.5. (Pole-free strip beneath the real axis.) *The operator family $R_\chi(k)$ defined above is holomorphic on $\Im k > 0$. Assume that Ω_i is star-shaped and the condition (3.1) is satisfied. Then, there exists $C_j > 0$, $j = 1, 2, 3$ such that $R_\chi(k)$ extends from the upper-half plane to a holomorphic operator family on $|\Re k| \geq C_1, \Im k \geq -C_2$ satisfying the estimate*

$$\|R_\chi(k)\|_{L^2(\Omega_i) \oplus L^2(\Omega_o) \rightarrow L^2(\Omega_i) \oplus L^2(\Omega_o)} \leq \frac{C_3}{|k|} \quad (3.6)$$

in this region.

This follows from the bound of Theorem 3.1 using the result [69, Lemma 2.3]. Recall that this result of Vodev takes a resolvent estimate on the real axis and converts it into a resolvent estimate in a strip beneath the real axis. In principle, one could go into the details of this result and make the width of the strip explicit in the constant from the bound on the real axis. Since Theorem 3.1 gives an explicit expression for that constant, we would then have an explicit lower bound for the width of the strip.

3.3 Discussion of the main results in the context of previous results

We now discuss Theorems 3.1–3.2 and 3.5 in the context of the results summarised in (a)–(e) of §1 (i.e. [5, 8–11, 28, 56, 57]) and other related work.

We focus on results about the Helmholtz transmission problem of Definition 2.4, i.e. one penetrable obstacle and piecewise-constant wavenumber. Many of these results apply to the more-general case when the wavenumber is piecewise-smooth, but we focus on the piecewise-constant case. There is also a substantial literature on the Helmholtz equation with continuous wavenumber, including [6] and [55]; for a survey of these result we refer the reader to [30, §2.4]. At the end of this section we briefly discuss (i) truncated Helmholtz transmission problems (in Remark 3.7), (ii) Helmholtz transmission problems with piecewise-constant wavenumber but *not* involving a bounded obstacle (in Remark 3.8), and (iii) Helmholtz transmission problems when $n_i \in \mathbb{C}$ with $\Im n_i > 0$ (in Remark 3.9).

The results summarised in (a)–(d) of §1 all consider the case when $a_i = a_o = n_o = A_D = 1$ and $g_D = g_N = 0$; that is, the BVP

$$\begin{aligned} \Delta u_i + k^2 n_i u_i &= f_i && \text{in } \Omega_i, \\ \Delta u_o + k^2 u_o &= f_o && \text{in } \Omega_o, \\ u_o &= u_i && \text{on } \Gamma, \\ \partial_{\mathbf{n}} u_o &= A_N \partial_{\mathbf{n}} u_i && \text{on } \Gamma, \\ u_o &\in \text{SRC}(k). \end{aligned} \quad (3.7)$$

The results summarised in (e) of §1 consider (3.7) with $A_N = 1$; these results are in a slightly different direction to those of (a)–(d) (involving bounds in different norms) and so we discuss them separately in Remark 3.6 below.

In this discussion we use the notation for the cut-off resolvent $R_\chi(k)$, and observe that the bound (3.2) in the case of the BVP (3.7) is essentially equivalent to the following bound: given $k_0 > 0$ there exist $C_m, m = 0, 1$, such that

$$\|R_\chi(k)\|_{L^2 \rightarrow H^m} \leq C_m k^{m-1} \quad \text{for all } k \geq k_0, \quad (3.8)$$

where L^2 and H^0 denote $L^2(\Omega_i) \oplus L^2(\Omega_o)$, H^1 denotes $H^1(\Omega_i) \oplus H^1(\Omega_o)$, and the constants C_m are given explicitly in terms of $n_i, A_N, d, \text{diam}(\Omega_i)$, and R (with R such that the support of the cut-off function χ_1 appearing in the definition of $R_\chi(k)$ (3.5) is contained in B_R). Once the bound

$$\|R_\chi(k)\|_{L^2 \rightarrow L^2} \leq C_0 k^{-1} \quad \text{for all } k \geq k_0, \quad (3.9)$$

is proven, Green's identity can be used to prove the $L^2 \rightarrow H^1$ bound, with the constant C_1 in (3.8) then given explicitly in terms of $n_i, A_N, d, \text{diam}(\Omega_i)$, R , and C_0 (the analogous argument for scattering by impenetrable Dirichlet or Neumann obstacles is given in, e.g., [61, Lemma 2.2]).

Cardoso, Popov, and Vodev [10] proved the bound (3.9) when Ω_i is a smooth, convex obstacle with strictly positive curvature; the existence of a resonance-free strip then followed from Vodev's result in [69]. The bound in [10] is proved under the assumptions $n_i < 1$ and $A_N > 0$ and the dependence of the constant C_0 on n_i and A_N is not given. The conditions $n_i < 1$ and $A_N > 0$ are less restrictive than our condition (3.1), which in this situation is $n_i \leq 1/A_N \leq 1$ (see Remark 5.1 below for how this condition appears in our proof).

The particular case when Ω_i is a ball shows that a strip is the largest region one can prove is free of resonances in the case $n_i < 1$ and $A_N > 0$. This was known in [56], but the recent results of Galkowski [28] in the case when Ω_i is C^∞ with strictly positive curvature include bounds on the width of the resonance-free strip in terms of appropriate averages of the reflectivity and chord lengths of the billiard ball trajectories in Ω_i [28, Theorem 1], and these bounds are sharp when Ω_i is a ball [28, §12]. Furthermore, these results (which build on earlier work by Cardoso, Popov, and Vodev [11]) show that if $n_i < 1$ and $\sqrt{n_i} < 1/A_N$, then the resonances themselves lie in a strip (i.e. there exist $C_1, C_2 > 0$ such that the resonances k satisfy $-C_2 \leq \Im k \leq -C_1$).

Popov and Vodev [57] showed that when $n_i > 1$ and $A_N > 0$ there exists a sequence of resonances tending to the real-axis and one has super-algebraic growth of $\|R_\chi(k)\|_{L^2 \rightarrow L^2}$ through a sequence of complex wavenumbers with super-algebraically small imaginary parts; we recap this result in more detail in §6.1. Exponential growth in k is the fastest growth possible by the results of Bellassoued [5]. Indeed, he proved that for any $n_i > 0$ and any $A_N > 0$ there exist $\tilde{C}_j, j = 1, \dots, 5$, such that

$$\|R_\chi(k)\|_{L^2 \rightarrow L^2} \leq \tilde{C}_1 \exp(\tilde{C}_2 |\Re k|)$$

in the region

$$\Re k \geq \tilde{C}_3, \quad \Im k \geq -\tilde{C}_4 \exp(-\tilde{C}_5 |\Re k|),$$

implying there is always an exponentially-small region free of resonances.

Remark 3.6. (Transmission problems when the obstacle is a 2- or 3-d ball.) When Ω_i is a ball, the solution of BVP (2.2) can be written explicitly using separation of variables and expansions in Fourier–Bessel functions. Capdeboscq and co-authors considered this problem for BVP (3.7) with $A_N = 1$ for $d = 2$ in [8] and for $d = 3$ in [9], with the main results summarised in [1, Chapter 5]. These results differ from the resolvent estimates discussed above, since they involve Sobolev norms of arbitrary order on spherical surfaces in D_R (hence the radial derivative term in the $H^1(D_R)$ norm is not directly controlled, nor the $H^1(\Omega_i)$ norm). Some of these results describe in detail the behaviour of the solution when $n_i > 1$, including the super-algebraic growth of the solution operator through a sequence of real wavenumbers, and we recap them in §6 below.

Remark 3.7. (Truncated transmission problems.) When solving scattering problems on unbounded domains numerically, it is common to truncate the domain and impose a boundary condition to approximate the Sommerfeld radiation condition; the simplest such boundary condition is the impedance condition $\partial_n u - iku = 0$ (see, e.g., the discussion in [4, §5.1] and the references therein). The truncated transmission problem is therefore equivalent to the interior impedance problem with piecewise-constant wavenumber. The paper [12] contains the analogue of the results in [10] for the truncated transmission problem (and in particular the bound (3.9) above). Wavenumber-explicit bounds on this BVP have recently been obtained in [3, 17, 30, 31, 59] for real n_i and [54] for complex n_i . Apart from [59], these recent investigations all use Morawetz identities (either explicitly or implicitly), with the impedance boundary condition dealt with as described in Remark 4.5 below (again,

either explicitly or implicitly). The investigation [59] concerns the interior impedance problem in 1-d with piecewise-constant wavenumber, and uses the fact that, in this case, the Green's function can be expressed in terms of the solution of a linear system.

Remark 3.8. (Transmission problems not involving a bounded obstacle.) Identities related to those of Morawetz have also been used to prove results about (i) scattering by rough surfaces when the wave-number is piecewise constant and (ii) the transmission problem through an infinite penetrable layer (where in both cases the wavenumbers satisfy appropriate analogues of (3.1)). For (i) see [13, 38, 70, 71], and [65, Chapter 2] (these works consider more general classes of wave-number that include piecewise-constant cases), and for (ii) see [16, 26, 43], and [65, Chapter 4]. The identities used are essentially (4.2) below with $\beta = 0$ and the vector field \mathbf{x} replaced by a vector field perpendicular to the surface/layer.

Remark 3.9. (Transmission problems when $\Im n_i > 0$.) Remark 5.2 below shows how analogues of Theorems 3.1–3.2 hold when $n_i \in \mathbb{C}$ with $0 < \Im n_i \leq \delta/k$ and δ is a sufficiently small constant (the occurrences of n_i in the conditions (3.1), (5.7), and (3.3) are then replaced by $\Re n_i$). This condition on the imaginary part is similar to that in [53], with this paper considering the Helmholtz transmission problem when Ω_i is the union of two concentric balls, modelling an inhomogeneity (with n_i real) surrounded by an absorbing layer (with n_i complex and $\Im n_i$ proportional to $1/k$). Like the works [1, 8, 9] discussed above, [53] is interested in bounding the solution away from Ω_i , but instead of using separation of variables, [53] uses Morawetz identities to prove its bounds. The paper [35] proves bounds analogous to those in [1, 8, 9] in the case when Ω_i is a ball and $\Im n_i > 0$, again using separation of variables and bounds on Bessel and Hankel functions.

4 Morawetz identities

In this section we prove the identities that are the basis of the proofs of Theorems 3.1 and 3.2. The history of these identities is briefly discussed in Remark 4.6 below.

Lemma 4.1. (Morawetz-type identity.) Let $D \subset \mathbb{R}^d$, $d \geq 2$. Let $v \in C^2(D)$ and let $a, n, \alpha, \beta \in \mathbb{R}$. Let

$$\mathcal{L}_{a,n}v := a\Delta v + k^2nv,$$

and let

$$\mathcal{M}v := \mathbf{x} \cdot \nabla v - ik\beta v + \alpha v. \quad (4.1)$$

Then

$$\begin{aligned} 2\Re\{\overline{\mathcal{M}v} \mathcal{L}_{a,n}v\} &= \nabla \cdot \left[2\Re\{\overline{\mathcal{M}v} a\nabla v\} + \mathbf{x}(k^2n|v|^2 - a|\nabla v|^2) \right] \\ &\quad - (2\alpha - d + 2)a|\nabla v|^2 - (d - 2\alpha)nk^2|v|^2. \end{aligned} \quad (4.2)$$

Proof. This follows from expanding the divergence on the right-hand side of (4.2). Note that the identity (4.2) is a special case of both [63, Lemma 2.1] (where the multiplier $\mathcal{M}u$ is generalised) and [30] (where the operator $\mathcal{L}_{a,n}$ is generalised). \square

The proofs of the main results are based on integrating the identity (4.2) over Ω_i and $D_R := \Omega_o \cap B_R$ and using the divergence theorem. Our next result, therefore, is an integrated version of (4.2). To state this result it is convenient to define the space

$$V(D) := \left\{ v \in H^1(D) : \Delta v \in L^2(D), \partial_{\boldsymbol{\nu}} v \in L^2(\partial D), v \in H^1(\partial D) \right\}, \quad (4.3)$$

where D is a bounded Lipschitz open set with outward-pointing unit normal vector $\boldsymbol{\nu}$.

Lemma 4.2. (Integrated form of the Morawetz identity (4.2).) Let D be a bounded Lipschitz open set, with boundary ∂D and outward-pointing unit normal vector $\boldsymbol{\nu}$. If $v \in V(D)$, $a, n, \alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned} &\int_D 2\Re\{\overline{\mathcal{M}v} \mathcal{L}_{a,n}v\} + (2\alpha - d + 2)a|\nabla v|^2 + (d - 2\alpha)nk^2|v|^2 \\ &= \int_{\partial D} (\mathbf{x} \cdot \boldsymbol{\nu}) \left(a|\partial_{\boldsymbol{\nu}} v|^2 - a|\nabla_T v|^2 + k^2n|v|^2 \right) + 2\Re\left\{ (\mathbf{x} \cdot \overline{\nabla_T v} + ik\beta\bar{v} + \alpha\bar{v})a\partial_{\boldsymbol{\nu}} v \right\}. \end{aligned} \quad (4.4)$$

Proof. If $v \in C^\infty(\overline{D})$, then (4.4) follows from divergence theorem $\int_D \nabla \cdot \mathbf{F} = \int_{\partial D} \mathbf{F} \cdot \boldsymbol{\nu}$. By [22, Lemmas 2 and 3], $C^\infty(\overline{D})$ is dense in $V(D)$ and the result then follows since (4.4) is continuous in v with respect to the topology of $V(D)$. \square

The proofs of the main results use different multipliers in different domains. More precisely, we use

$$\begin{aligned} A_D A_N \left(\mathbf{x} \cdot \nabla u - i k R \sqrt{\frac{n_o}{a_o}} u + \frac{d-1}{2} u \right) & \quad \text{in } \Omega_i, \\ \mathbf{x} \cdot \nabla u - i k R \sqrt{\frac{n_o}{a_o}} u + \frac{d-1}{2} u & \quad \text{in } D_R, \text{ and} \\ \mathbf{x} \cdot \nabla u - i k r \sqrt{\frac{n_o}{a_o}} u + \frac{d-1}{2} u & \quad \text{in } \mathbb{R}^d \setminus D_R, \end{aligned}$$

where R is the radius of the ball in which we bound the solution, and $r := |\mathbf{x}|$. The first two of these three multipliers are multiples of $\mathcal{M}u$, as defined in (4.1), with $\alpha = \frac{d-1}{2}$ and $\beta = R\sqrt{n_o/a_o}$; the third one is slightly different in that the coefficient $\beta = r\sqrt{n_o/a_o}$ depends on the position vector. Therefore, the identity arising from this last multiplier is not covered by Lemma 4.1 but is given in the following lemma.

Lemma 4.3. (Morawetz–Ludwig identity, [51, Equation 1.2].) *Let $v \in C^2(D)$ for some $D \subset \mathbb{R}^d$, $d \geq 2$. Let $\kappa \in \mathbb{R}$, $\mathcal{L}v := (\Delta + \kappa^2)v$ and let*

$$\mathcal{M}_\alpha v := r \left(v_r - i\kappa v + \frac{\alpha}{r} v \right), \quad (4.5)$$

where $\alpha \in \mathbb{R}$ and $v_r = \mathbf{x} \cdot \nabla v / r$. Then

$$\begin{aligned} 2\Re\{\overline{\mathcal{M}_\alpha v} \mathcal{L}v\} &= \nabla \cdot \left[2\Re\{\overline{\mathcal{M}_\alpha v} \nabla v\} + (\kappa^2 |v|^2 - |\nabla v|^2) \mathbf{x} \right] \\ &+ (2\alpha - (d-1))(\kappa^2 |v|^2 - |\nabla v|^2) - (|\nabla v|^2 - |v_r|^2) - |v_r - i\kappa v|^2. \end{aligned} \quad (4.6)$$

The Morawetz–Ludwig identity (4.6) is a variant of the identity (4.2) with $a = 1$, $n = 1$, $k = \kappa$, and $\beta = r$ (instead of being a constant); for a proof, see [51], [62, Proof of Lemma 2.2], or [63, Proof of Lemma 2.3].

As stated above, we use the Morawetz–Ludwig identity in $\mathbb{R}^d \setminus B_R$ (it turns out that this identity “takes care” of the contribution from infinity). It is convenient to encode the application of this identity in $\mathbb{R}^d \setminus B_R$ in the following lemma (slightly more general versions of which appear in [15, Lemma 2.1] and [30]).

Lemma 4.4. (Inequality on Γ_R used to deal with the contribution from infinity.) *Let u be a solution of the homogeneous Helmholtz equation $\mathcal{L}u = 0$ in $\mathbb{R}^d \setminus \overline{B_{R_0}}$ (with $d \geq 2$), for some $R_0 > 0$, satisfying the Sommerfeld radiation condition. Then, for $R > R_0$,*

$$\int_{\Gamma_R} R \left(\left| \frac{\partial u}{\partial r} \right|^2 - |\nabla_T u|^2 + \kappa^2 |u|^2 \right) - 2\kappa R \Im \int_{\Gamma_R} \bar{u} \frac{\partial u}{\partial r} + (d-1) \Re \int_{\Gamma_R} \bar{u} \frac{\partial u}{\partial r} \leq 0, \quad (4.7)$$

where ∇_T is the tangential gradient on $r = R$ (recall that this is such that $\nabla v = \nabla_T v + \hat{\mathbf{x}} v_r$ on $r = R$).

Proof. We now integrate (4.6) with $v = u$ and $2\alpha = d-1$ over $B_{R_1} \setminus B_R$, use the divergence theorem, and then let $R_1 \rightarrow \infty$ (note that using the divergence theorem is allowed since u is C^∞ by elliptic regularity).

Writing the identity (4.6) as $\nabla \cdot \mathbf{Q}(v) = P(v)$, we have that if u is a solution of $\mathcal{L}u = 0$ in $\mathbb{R}^d \setminus \overline{B_{R_1}}$ satisfying the Sommerfeld radiation condition (2.1), then

$$\int_{\Gamma_{R_1}} \mathbf{Q}(u) \cdot \hat{\mathbf{x}} \rightarrow 0 \quad \text{as } R_1 \rightarrow \infty$$

(independent of the value of α in the multiplier $\mathcal{M}_\alpha u$); see [51, Proof of Lemma 5], [62, Lemma 2.4]. Note that, although the set-up of [62] is for $d = 2, 3$, the proof of [62, Lemma 2.4] holds for $d \geq 2$.

Then, using the decomposition $\nabla v = \nabla_T v + \widehat{\mathbf{x}} v_r$ on the integral over Γ_R (or equivalently the right-hand side of (4.4)), we obtain that

$$\begin{aligned} \int_{\Gamma_R} \mathbf{Q}(u) \cdot \widehat{\mathbf{x}} &= \int_{\Gamma_R} R \left(\left| \frac{\partial u}{\partial r} \right|^2 - |\nabla_T u|^2 + \kappa^2 |u|^2 \right) - 2\kappa R \Im \int_{\Gamma_R} \bar{u} \frac{\partial u}{\partial r} + (d-1) \Re \int_{\Gamma_R} \bar{u} \frac{\partial u}{\partial r} \\ &= - \int_{\mathbb{R}^d \setminus B_R} \left((|\nabla u|^2 - |u_r|^2) + |u_r - i\kappa u|^2 \right) \leq 0. \end{aligned}$$

□

Remark 4.5. (Far-field impedance boundary condition.) *If the infinite domain Ω_o is truncated, and the radiation condition approximated by an impedance boundary condition, then an analogous inequality to that in Lemma 4.4 holds; see [30, Lemma 4.6]. This analogous inequality allows one to extend the results of Theorems 3.1 and 3.2 to this truncated BVP (as mentioned in Remark 3.4 above); see [30] for more details.*

Remark 4.6. (Bibliographic remarks.) *The multiplier $\mathbf{x} \cdot \nabla v$ was introduced by Rellich in [58], and has been well-used since then in the study of the Laplace, Helmholtz, and other elliptic equations, see, e.g., the references in [14, §5.3], [48, §1.4].*

The idea of using a multiplier that is a linear combination of derivatives of v and v itself, such as $\mathcal{M}v$, is attributed by Morawetz in [49] to Friedrichs. The multiplier $\mathcal{M}_{\alpha} v$ (4.5) for the Helmholtz equation was introduced by Morawetz and Ludwig in [51] and the multiplier $\mathcal{M}v$ (4.1) (with \mathbf{x} replaced by a general vector field and α and β replaced by general scalar fields) is implicit in Morawetz's paper [50] (for more discussion on this, see [63, Remark 2.7]).

5 Proofs of the main results

Proof of Theorem 3.1. We use the integrated Morawetz identity (4.4) with first $D = \Omega_i$ and then $D = D_R$. In both cases we take $2\alpha = d - 1$ and use the same (as yet unspecified) β ; in the first case we take $v = u_i$, $a = a_i$, $n = n_i$, and in the second case we take $v = u_o$, $a = a_o$, $n = n_o$. Using (4.4) is justified since the regularity result in Lemma 2.7 shows that $u_i \in V(\Omega_i)$ and $u_o \in V(D_R)$. We get

$$\begin{aligned} &\int_{\Omega_i} a_i |\nabla u_i|^2 + n_i k^2 |u_i|^2 \\ &= -2\Re \int_{\Omega_i} \overline{\mathcal{M}u_i} f_i + \int_{\Gamma} (\mathbf{x} \cdot \mathbf{n}) \left(a_i |\partial_{\mathbf{n}} u_i|^2 - a_i |\nabla_T u_i|^2 + k^2 n_i |u_i|^2 \right) \\ &\quad + 2\Re \left\{ \left(\mathbf{x} \cdot \overline{\nabla_T u_i} + ik\beta \overline{u_i} + \frac{d-1}{2} \overline{u_i} \right) a_i \partial_{\mathbf{n}} u_i \right\} \end{aligned} \quad (5.1)$$

and

$$\begin{aligned} &\int_{D_R} a_o |\nabla u_o|^2 + n_o k^2 |u_o|^2 \\ &= -2\Re \int_{D_R} \overline{\mathcal{M}u_o} f_o - \int_{\Gamma} (\mathbf{x} \cdot \mathbf{n}) \left(a_o |\partial_{\mathbf{n}} u_o|^2 - a_o |\nabla_T u_o|^2 + k^2 n_o |u_o|^2 \right) \\ &\quad + 2\Re \left\{ \left(\mathbf{x} \cdot \overline{\nabla_T u_o} + ik\beta \overline{u_o} + \frac{d-1}{2} \overline{u_o} \right) a_o \partial_{\mathbf{n}} u_o \right\} \\ &\quad + \int_{\Gamma_R} R \left(a_o \left| \frac{\partial u_o}{\partial r} \right|^2 - a_o |\nabla_T u_o|^2 + k^2 n_o |u_o|^2 \right) \\ &\quad - 2a_o k \beta \Im \int_{\Gamma_R} \overline{u_o} \frac{\partial u_o}{\partial r} + a_o (d-1) \Re \int_{\Gamma_R} \overline{u_o} \frac{\partial u_o}{\partial r}. \end{aligned} \quad (5.2)$$

Multiplying the inequality (4.7) by a_o and letting $\kappa = k\sqrt{n_o/a_o}$, we see that if we choose $\beta = R\sqrt{n_o/a_o}$ then the terms on Γ_R on the right-hand side of (5.2) are non-positive.

We then multiply (5.2) by an arbitrary $\eta > 0$ and add to (5.1) to get

$$\int_{\Omega_i} (a_i |\nabla u_i|^2 + k^2 n_i |u_i|^2) + \eta \int_{D_R} (a_o |\nabla u_o|^2 + k^2 n_o |u_o|^2)$$

$$\begin{aligned}
&\leq -2\Re \int_{\Omega_i} \overline{\mathcal{M}u_i} f_i - 2\eta\Re \int_{D_R} \overline{\mathcal{M}u_o} f_o \\
&+ \int_{\Gamma} (\mathbf{x} \cdot \mathbf{n}) \left(a_i |\partial_{\mathbf{n}} u_i|^2 - \eta a_o |\partial_{\mathbf{n}} u_o|^2 - a_i |\nabla_T u_i|^2 + \eta a_o |\nabla_T u_o|^2 + k^2 n_i |u_i|^2 - \eta k^2 n_o |u_o|^2 \right) \\
&+ 2\Re \left\{ \left(\mathbf{x} \cdot \overline{\nabla_T u_i} + ik \sqrt{\frac{n_o}{a_o}} R \overline{u_i} + \frac{d-1}{2} \overline{u_i} \right) a_i \partial_{\mathbf{n}} u_i \right\} \\
&- 2\eta\Re \left\{ \left(\mathbf{x} \cdot \overline{\nabla_T u_o} + ik \sqrt{\frac{n_o}{a_o}} R \overline{u_o} + \frac{d-1}{2} \overline{u_o} \right) a_o \partial_{\mathbf{n}} u_o \right\}. \tag{5.3}
\end{aligned}$$

The volume terms at the right-hand side are bounded above by

$$\begin{aligned}
&\|f_i\|_{\Omega_i} \left(2 \operatorname{diam}(\Omega_i) \|\nabla u_i\|_{\Omega_i} + \left(2k \sqrt{\frac{n_o}{a_o}} R + d-1 \right) \|u_i\|_{\Omega_i} \right) \\
&+ \eta \|f_o\|_{D_R} \left(2R \|\nabla u_o\|_{D_R} + \left(2k \sqrt{\frac{n_o}{a_o}} R + d-1 \right) \|u_o\|_{\Omega_o} \right) \tag{5.4}
\end{aligned}$$

(recall that $\|\cdot\|_{\Omega_{i/o}}$ denotes the L^2 norm on $\Omega_{i/o}$). We now focus on the terms on Γ , and recall that we are assuming that $g_D = g_N = 0$. Our goal is to choose η so that the terms without a sign (i.e. those on the last two lines of (5.3)) cancel. Using the transmission conditions in (2.2), we see that this cancellation occurs if $\eta = 1/(A_D A_N)$. (It is at this point that we need A_D and A_N to be real; indeed, if the product $A_D A_N$ has non-zero real and imaginary parts, we cannot choose even a complex η to cancel these terms).

Making this choice of η and using the transmission conditions, we see that the remaining terms on Γ become

$$\int_{\Gamma} (\mathbf{x} \cdot \mathbf{n}) \left(k^2 n_i |u_i|^2 \left(1 - \frac{A_D}{A_N} \frac{n_o}{n_i} \right) + a_i |\partial_{\mathbf{n}} u_i|^2 \left(1 - \frac{A_N}{A_D} \frac{a_i}{a_o} \right) + a_i |\nabla_T u_i|^2 \left(-1 + \frac{A_D}{A_N} \frac{a_o}{a_i} \right) \right). \tag{5.5}$$

These terms are negative and thus can be neglected if

$$A_N n_i \leq A_D n_o \quad \text{and} \quad A_N a_i \geq A_D a_o,$$

or equivalently if (3.1) holds.

In summary, under the conditions (3.1), we have that

$$a_i \|\nabla u_i\|_{\Omega_i}^2 + k^2 n_i \|u_i\|_{\Omega_i}^2 + \frac{1}{A_D A_N} \left(a_o \|\nabla u_o\|_{D_R}^2 + k^2 n_o \|u_o\|_{D_R}^2 \right)$$

is bounded by (5.4). Using the Cauchy–Schwarz and Young inequalities we obtain the assertion of Theorem 3.1. \square

Remark 5.1. (The origin of the condition (3.1).) Condition (3.1) comes from requiring that each of the terms in (5.5) are non-positive. These terms are not independent, however, since they all depend on u_i . Despite this connection, we have not been able to lessen the requirements of (3.1) using only these elementary arguments, other than in Proposition 5.3 below where the term on Γ involving $|u_i|^2$ is controlled by the norms of u_i in Ω_i via a trace inequality.

Proof of Theorem 3.2. In the proof of Theorem 3.1, the assumption $g_D = g_N = 0$ was used to derive (5.5) from (5.3). Now that g_D and g_N are not necessarily zero, we expand the terms on Γ appearing in (5.3) using the transmission conditions with $g_D, g_N \neq 0$ and the fact that $\eta = 1/A_D A_N$. We control these terms on Γ using $|\mathbf{x}| \leq \operatorname{diam}(\Omega_i)$ and $-\mathbf{x} \cdot \mathbf{n} \leq -\gamma \operatorname{diam}(\Omega_i)$ for a.e. $\mathbf{x} \in \Gamma$, and $(a_i A_N - a_o A_D) > 0$ and $(n_o A_D - n_i A_N) > 0$ from assumption (3.3). We apply the weighted Young’s inequality to nine terms, denoted T_1, \dots, T_9 , using positive coefficients ξ_1, \dots, ξ_9 :

$$\begin{aligned}
&\int_{\Gamma} (\mathbf{x} \cdot \mathbf{n}) \left(a_i |\partial_{\mathbf{n}} u_i|^2 - a_i |\nabla_T u_i|^2 + k^2 n_i |u_i|^2 \right. \\
&- \frac{1}{a_o A_D A_N} |A_N a_i \partial_{\mathbf{n}} u_i + g_N|^2 + \frac{a_o}{A_D A_N} |A_D \nabla_T u_i + \nabla_T g_D|^2 - k^2 \frac{n_o}{A_D A_N} |A_D u_i + g_D|^2 \Big) \\
&+ 2\Re \left\{ \left(\mathbf{x} \cdot \overline{\nabla_T u_i} + ik \sqrt{\frac{n_o}{a_o}} R \overline{u_i} + \frac{d-1}{2} \overline{u_i} \right) a_i \partial_{\mathbf{n}} u_i \right\}
\end{aligned}$$

$$\begin{aligned}
& -2\Re \left\{ \left(\mathbf{x} \cdot \left(\nabla_T u_i + \frac{1}{A_D} \nabla_T g_D \right) + \left(ik\sqrt{\frac{n_o}{a_o}} R + \frac{d-1}{2} \right) \left(u_i + \frac{1}{A_D} g_D \right) \right) \left(a_i \partial_{\mathbf{n}} u_i + \frac{1}{A_N} g_N \right) \right\} \\
& = \int_{\Gamma} (\mathbf{x} \cdot \mathbf{n}) \left(a_i \left(1 - \frac{a_i A_N}{a_o A_D} \right) |\partial_{\mathbf{n}} u_i|^2 + a_i \left(\frac{a_o A_D}{a_i A_N} - 1 \right) |\nabla_T u_i|^2 + k^2 n_i \left(1 - \frac{n_o A_D}{n_i A_N} \right) |u_i|^2 \right. \\
& \quad - \frac{1}{a_o A_D A_N} |g_N|^2 + \frac{a_o}{A_D A_N} |\nabla_T g_D|^2 - k^2 \frac{n_o}{A_D A_N} |g_D|^2 \\
& \quad - 2 \frac{a_i}{a_o A_D} \underbrace{\Re \{ \partial_{\mathbf{n}} u_i \overline{g_N} \}}_{T_1} + 2 \frac{a_o}{A_N} \underbrace{\Re \{ \nabla_T u_i \cdot \nabla_T \overline{g_D} \}}_{T_2} - 2k^2 \frac{n_o}{A_N} \underbrace{\Re \{ u_i \overline{g_D} \}}_{T_3} \Bigg) \\
& \quad - 2 \frac{1}{A_N} \underbrace{\Re \{ \mathbf{x} \cdot \nabla_T \overline{u_i} g_N \}}_{T_4} - 2 \frac{1}{A_N} \underbrace{\Re \left\{ \left(ik\sqrt{\frac{n_o}{a_o}} R + \frac{d-1}{2} \right) \overline{u_i} g_N \right\}}_{T_5} \\
& \quad - 2 \frac{a_i}{A_D} \underbrace{\Re \{ \mathbf{x} \cdot \nabla_T \overline{g_D} \partial_{\mathbf{n}} u_i \}}_{T_6} - 2 \frac{a_i}{A_D} \underbrace{\Re \left\{ \left(ik\sqrt{\frac{n_o}{a_o}} R + \frac{d-1}{2} \right) \overline{g_D} \partial_{\mathbf{n}} u_i \right\}}_{T_7} \\
& \quad - 2 \frac{1}{A_N A_D} \underbrace{\Re \{ \mathbf{x} \cdot \nabla_T \overline{g_D} g_N \}}_{T_8} - 2 \frac{1}{A_N A_D} \underbrace{\Re \left\{ \left(ik\sqrt{\frac{n_o}{a_o}} R + \frac{d-1}{2} \right) \overline{g_D} g_N \right\}}_{T_9} \\
& \leq -\gamma \text{diam}(\Omega_i) \left(a_i \frac{a_i A_N - a_o A_D}{a_o A_D} \|\partial_{\mathbf{n}} u_i\|_{\Gamma}^2 \right. \\
& \quad \left. + \frac{a_i A_N - a_o A_D}{A_N} \|\nabla_T u_i\|_{\Gamma}^2 + k^2 \frac{n_o A_D - n_i A_N}{A_N} \|u_i\|_{\Gamma}^2 \right) \\
& \quad - \gamma \text{diam}(\Omega_i) \frac{1}{a_o A_D A_N} \|g_N\|_{\Gamma}^2 + \text{diam}(\Omega_i) \frac{a_o}{A_D A_N} \|\nabla_T g_D\|_{\Gamma}^2 - k^2 \gamma \text{diam}(\Omega_i) \frac{n_o}{A_D A_N} \|g_D\|_{\Gamma}^2 \\
& \quad + \xi_1 a_i \gamma \text{diam}(\Omega_i) \frac{a_i A_N - a_o A_D}{a_o A_D} \|\partial_{\mathbf{n}} u_i\|_{\Gamma}^2 + \frac{a_i \text{diam}(\Omega_i)}{\xi_1 \gamma a_o A_D (a_i A_N - a_o A_D)} \|g_N\|_{\Gamma}^2 \\
& \quad + \xi_2 \gamma \text{diam}(\Omega_i) \frac{a_i A_N - a_o A_D}{A_N} \|\nabla_T u_i\|_{\Gamma}^2 + \frac{\text{diam}(\Omega_i) a_o^2}{\xi_2 \gamma A_N (a_i A_N - a_o A_D)} \|\nabla_T g_D\|_{\Gamma}^2 \\
& \quad + \xi_3 k^2 \gamma \text{diam}(\Omega_i) \frac{n_o A_D - n_i A_N}{A_N} \|u_i\|_{\Gamma}^2 + \frac{k^2 \text{diam}(\Omega_i) n_o^2}{\xi_3 \gamma A_N (n_o A_D - n_i A_N)} \|g_D\|_{\Gamma}^2 \\
& \quad + \xi_4 \gamma \text{diam}(\Omega_i) \frac{a_i A_N - a_o A_D}{A_N} \|\nabla_T u_i\|_{\Gamma}^2 + \frac{\text{diam}(\Omega_i)}{\xi_4 \gamma A_N (a_i A_N - a_o A_D)} \|g_N\|_{\Gamma}^2 \\
& \quad + \xi_5 k^2 \gamma \text{diam}(\Omega_i) \frac{n_o A_D - n_i A_N}{A_N} \|u_i\|_{\Gamma}^2 + \frac{\left| ik\sqrt{\frac{n_o}{a_o}} R + \frac{d-1}{2} \right|^2}{\xi_5 \gamma \text{diam}(\Omega_i) k^2 A_N (n_o A_D - n_i A_N)} \|g_N\|_{\Gamma}^2 \\
& \quad + \xi_6 a_i \gamma \text{diam}(\Omega_i) \frac{a_i A_N - a_o A_D}{a_o A_D} \|\partial_{\mathbf{n}} u_i\|_{\Gamma}^2 + \frac{a_i a_o \text{diam}(\Omega_i)}{\xi_6 \gamma A_D (a_i A_N - a_o A_D)} \|\nabla_T g_D\|_{\Gamma}^2 \\
& \quad + \xi_7 a_i \gamma \text{diam}(\Omega_i) \frac{a_i A_N - a_o A_D}{a_o A_D} \|\partial_{\mathbf{n}} u_i\|_{\Gamma}^2 + \frac{a_i a_o \left| ik\sqrt{\frac{n_o}{a_o}} R + \frac{d-1}{2} \right|^2}{\xi_7 \gamma A_D \text{diam}(\Omega_i) (a_i A_N - a_o A_D)} \|g_D\|_{\Gamma}^2 \\
& \quad + \xi_8 \frac{\text{diam}(\Omega_i)}{a_o A_N A_D} \|g_N\|_{\Gamma}^2 + \frac{\text{diam}(\Omega_i) a_o}{\xi_8 A_N A_D} \|\nabla_T g_D\|_{\Gamma}^2 \\
& \quad + \xi_9 \frac{\text{diam}(\Omega_i)}{a_o A_N A_D} \|g_N\|_{\Gamma}^2 + \frac{a_o \left| ik\sqrt{\frac{n_o}{a_o}} R + \frac{d-1}{2} \right|^2}{\xi_9 \text{diam}(\Omega_i) A_D A_N} \|g_D\|_{\Gamma}^2.
\end{aligned}$$

We choose the weights as

$$\xi_1 = \xi_6 = \xi_7 = \frac{1}{3}, \quad \xi_2 = \xi_3 = \xi_4 = \xi_5 = \frac{1}{2}, \quad \xi_8 = \xi_9 = 1,$$

so that all terms containing u_i cancel each other, and we are left with

$$\frac{\text{diam}(\Omega_i)}{A_N} \left[-\frac{\gamma n_o}{A_D} + \frac{2n_o^2}{\gamma(n_o A_D - n_i A_N)} \right]$$

$$\begin{aligned}
& + \frac{1}{A_D \operatorname{diam}(\Omega_i)^2} \left(1 + \frac{3a_i A_N}{\gamma(a_i A_N - a_o A_D)} \right) \left(n_o R^2 + \frac{a_o(d-1)^2}{4k^2} \right) \Big] k^2 \|g_D\|_\Gamma^2 \\
& + \frac{\operatorname{diam}(\Omega_i) a_o}{A_D A_N} \left[2 + \frac{2a_o A_D + 3a_i A_N}{\gamma(a_i A_N - a_o A_D)} \right] \|\nabla_T g_D\|_\Gamma^2 \\
& + \frac{\operatorname{diam}(\Omega_i)}{a_o A_N A_D} \left[2 - \gamma + \frac{3a_i A_N + 2a_o A_D}{\gamma(a_i A_N - a_o A_D)} + \frac{2A_D \left(n_o R^2 + \frac{a_o(d-1)^2}{4k^2} \right)}{\gamma \operatorname{diam}(\Omega_i)^2 (n_o A_D - n_i A_N)} \right] \|g_N\|_\Gamma^2 \\
& \leq \left[\frac{2 \operatorname{diam}(\Omega_i) n_o^2}{\gamma A_N (n_o A_D - n_i A_N)} + \frac{(3 + \gamma) a_i \left(n_o R^2 + \frac{a_o(d-1)^2}{4k^2} \right)}{A_D \gamma \operatorname{diam}(\Omega_i) (a_i A_N - a_o A_D)} \right] k^2 \|g_D\|_\Gamma^2 \\
& + \frac{\operatorname{diam}(\Omega_i) a_o ((3 + 2\gamma) a_i A_N + 2a_o A_D)}{A_D A_N \gamma (a_i A_N - a_o A_D)} \|\nabla_T g_D\|_\Gamma^2 \\
& + \frac{1}{\gamma a_o A_N A_D} \left[\frac{\operatorname{diam}(\Omega_i) (4a_i A_N + 2a_o A_D)}{a_i A_N - a_o A_D} + \frac{2A_D \left(n_o R^2 + \frac{a_o(d-1)^2}{4k^2} \right)}{\operatorname{diam}(\Omega_i) (n_o A_D - n_i A_N)} \right] \|g_N\|_\Gamma^2,
\end{aligned}$$

where we used also $0 < \gamma \leq 1/2$ and dropped some negative terms. Then the bound in the assertion, (3.4), follows as in the proof of Theorem 3.1, recalling that the use of Young's inequality for the volume norms gives a further factor of 2 in front of norms on Γ in the final bound. \square

Remark 5.2. (Extensions of Theorems 3.1–3.2 to the case when $\Im n_i > 0$.) We now explain how analogues of Theorems 3.1–3.2 hold when $n_i \in \mathbb{C}$ with $0 < \Im n_i \leq \delta/k$ and δ is sufficiently small (the occurrences of n_i in the conditions (3.1), (5.7), and (3.3) are then replaced by $\Re n_i$). We first consider Theorem 3.1. Under the assumption that the existence, uniqueness, and regularity results of Lemma 2.7 hold for the boundary value problem with such n_i , Equation (5.3) now holds with n_i replaced by $\Re n_i$ and f_i replaced by $f_i - ik^2(\Im n_i)u_i$. We therefore have the extra term

$$-2\Im \int_{\Omega_i} \overline{\mathcal{M}u_i} k^2 (\Im n_i) u_i$$

on the right-hand side of (5.3). If $0 < \Im n_i \leq \delta/k$ and δ is sufficiently small, then this term can be absorbed into the weighted H^1 -norm of u_i on the left-hand side of (5.3) (using the Cauchy–Schwarz and weighted Young inequalities). The result is a bound with the same k -dependence as (3.2), but slightly different constants on the right-hand side. This proof of an a priori bound, under the assumption of existence, implies uniqueness. One can then check that the proof of existence and regularity in Appendix A goes through with n_i of this particular form. Finally, the extensions to the proof of Theorem 3.1 needed to prove Theorem 3.2 go through as before (since these only involve the terms on Γ).

Proof of Theorem 3.5. The result [69, Lemma 2.3] implies that the assertion of the theorem will hold if (i) $R_\chi(k)$ is holomorphic for $\Im k > 0$ and (ii) there exist $C_4 > 0$ and $k_0 > 0$ such that

$$\|R_\chi(k)\|_{L^2(\Omega_i) \oplus L^2(\Omega_o) \rightarrow L^2(\Omega_i) \oplus L^2(\Omega_o)} \leq \frac{C_4}{k} \quad \text{for all } k \geq k_0. \quad (5.6)$$

Note that, in applying Vodev's result, we take Vodev's obstacle Ω to be the empty set, $N_0 = 1$, Ω_1 equal to our Ω_i , and $g_{ij}^{(1)} = \delta_{ij}$. We also note that the set-up in [69] assumes that Ω_i is smooth. Nevertheless, the result [69, Lemma 2.3] boils down to a perturbation argument (via Neumann series) and a result about the free resolvent (i.e. the inverse of the Helmholtz operator in the absence of any obstacle) [69, Lemma 2.2]; both of these results are independent of Ω_i , and so [69, Lemma 2.3] is valid when Ω_i is Lipschitz.

Since Ω_i is star-shaped and the condition (3.1) is satisfied, Theorem 3.1 implies that the estimate on the real axis (5.6) holds, and thus we need only show that $R_\chi(k)$ is holomorphic on $\Im k > 0$.

Observe that $R_\chi(k)$ is well-defined for $\Im k \geq 0$ by the existence and uniqueness results of Lemma 2.7. Analyticity follows by applying the Cauchy–Riemann operator $\partial/\partial \bar{k}$ to the BVP (2.2). Indeed, by using Green's integral representation in $\Omega_{i/o}$ we find that $\partial(\Delta u)/\partial \bar{k} = \Delta(\partial u/\partial \bar{k})$, and similarly for ∇u . These in turn imply that $\partial_{\mathbf{n}}(\partial u/\partial \bar{k}) = \partial(\partial_{\mathbf{n}} u)/\partial \bar{k}$. Therefore, applying $\partial/\partial \bar{k}$ to (2.2), we find that $\partial u/\partial \bar{k}$ satisfies the Helmholtz transmission problem with zero volume and boundary data, and thus must vanish by the uniqueness result. \square

The condition (3.1) in Theorem 3.1 implies that $n_i/a_i \leq n_o/a_o$, namely that the wavelength $\lambda = (2\pi\sqrt{a})/(\sqrt{n}k)$ of the solution u is larger in the inner domain Ω_i than in Ω_o . In the next proposition we extend the result of Theorem 3.1 to a case where this condition is slightly violated.

Proposition 5.3. *Assume that Ω_i is star-shaped,*

$$\frac{n_o}{n_i} \left(\frac{n_i}{n_o} - \frac{A_D}{A_N} \right) \left(d + \sqrt{d^2 + \frac{4n_i}{a_i} (k \operatorname{diam}(\Omega_i))^2} \right) < 1, \quad \frac{A_D}{A_N} \leq \frac{a_i}{a_o}, \quad (5.7)$$

$g_N = g_D = 0$ and $k > 0$. Then the solution of BVP (2.2) satisfies

$$\begin{aligned} G \left(a_i \|\nabla u_i\|_{\Omega_i}^2 + k^2 n_i \|u_i\|_{\Omega_i}^2 \right) + \frac{1}{A_D A_N} \left(a_o \|\nabla u_o\|_{D_R}^2 + k^2 n_o \|u_o\|_{D_R}^2 \right) \\ \leq \left[\frac{4 \operatorname{diam}(\Omega_i)^2}{a_i} + \frac{1}{n_i} \left(2\sqrt{\frac{n_o}{a_o}} R + \frac{d-1}{k} \right)^2 \right] \|f_i\|_{\Omega_i}^2 \\ + \frac{1}{A_D A_N} \left[\frac{4R^2}{a_o} + \frac{1}{n_o} \left(2\sqrt{\frac{n_o}{a_o}} R + \frac{d-1}{k} \right)^2 \right] \|f_o\|_{D_R}^2, \end{aligned} \quad (5.8)$$

where G is defined by

$$G := \frac{1}{2} \left(1 - \frac{n_o}{n_i} \left(\frac{n_i}{n_o} - \frac{A_D}{A_N} \right) \left(d + \sqrt{d^2 + \frac{4n_i}{a_i} (k \operatorname{diam}(\Omega_i))^2} \right) \right) \quad (5.9)$$

and is positive by the first inequality in (5.7).

To better understand the condition (5.7), consider the simple case when $n_i = a_i = a_o = A_D = A_N = 1$. Then the condition (5.7) is satisfied if

$$n_o > 1 - \frac{1}{d + \sqrt{d^2 + 4(k \operatorname{diam}(\Omega_i))^2}}. \quad (5.10)$$

For a fixed $n_o > 1$ and sufficiently small, the condition (5.10) is an upper bound on k under which the estimate (5.8) holds—this is consistent with the results on super-algebraic growth in k of the resolvent for $n_o > 1$ recapped in §6 below. If we allow n_o to be a function of k , then the condition (5.10) implies that the estimate (5.8) holds if the distance between n_o and $n_i (= 1)$ decreases like $1/k$ as $k \rightarrow \infty$.

Proof of Proposition 5.3. The proof proceeds exactly the same was as the proof of Theorem 3.1 up to (5.5). Now, the assumption in (5.7) that $A_D a_o \leq A_N a_i$ implies that the terms in (5.5) are bounded by

$$\int_{\Gamma} (\mathbf{x} \cdot \mathbf{n}) \left(k^2 n_i |u_i|^2 \left(1 - \frac{A_D}{A_N} \frac{n_o}{n_i} \right) \right). \quad (5.11)$$

For all $w \in H^1(\Omega_i)$ and $\epsilon > 0$, we have the following weighted trace inequality:

$$\begin{aligned} \int_{\Gamma} (\mathbf{x} \cdot \mathbf{n}) |w|^2 &= \int_{\Omega_i} \operatorname{div}(\mathbf{x} |w|^2) \\ &= \int_{\Omega_i} (d|w|^2 + 2\mathbf{x} \cdot \Re\{w \nabla \bar{w}\}) \leq (d + \epsilon) \|w\|_{\Omega_i}^2 + \frac{1}{\epsilon} \operatorname{diam}(\Omega_i)^2 \|\nabla w\|_{\Omega_i}^2. \end{aligned} \quad (5.12)$$

We chose ϵ so that

$$\frac{d + \epsilon}{n_i k^2} = \frac{1}{\epsilon} \frac{\operatorname{diam}(\Omega_i)^2}{a_i}, \quad (5.13)$$

so that the right-hand side of (5.12) becomes

$$\frac{d + \epsilon}{n_i k^2} \left(n_i k^2 \|w\|_{\Omega_i}^2 + a_i \|\nabla w\|_{\Omega_i}^2 \right).$$

The requirement (5.13), and the fact that $\epsilon > 0$, imply that

$$\epsilon = \frac{1}{2} \left(-d + \sqrt{d^2 + \frac{4n_i}{a_i} (k \operatorname{diam}(\Omega_i))^2} \right).$$

We then get that (5.11) is bounded by

$$\frac{n_o}{n_i} \left(\frac{n_i}{n_o} - \frac{A_D}{A_N} \right) \frac{1}{2} \left(d + \sqrt{d^2 + \frac{4n_i}{a_i} (k \operatorname{diam}(\Omega_i))^2} \right) (n_i k^2 \|u_i\|_{\Omega_i}^2 + a_i \|\nabla u_i\|_{\Omega_i}^2).$$

The requirement (5.7) implies that this term is strictly less than $(n_i k^2 \|u_i\|_{\Omega_i}^2 + a_i \|\nabla u_i\|_{\Omega_i}^2)/2$, and thus the argument proceeds as before (with the other half of the norm being used to deal with the terms in (5.4) via the Cauchy–Schwarz and Young inequalities). \square

6 Super-algebraic growth in k when the condition (3.1) is violated

In §6.1 we adapt the results of [57], for C^∞ convex Ω_i , on super-algebraic growth of the solution operator through a sequence of complex wavenumbers when the condition (3.1) does not hold to prove an analogous result for real wavenumbers. In §6.2 we briefly highlight the analogous results of [1, 8, 9], valid for real wavenumbers, when Ω_i is a ball. As explained at the end of §1, our main motivation for doing this is the recent interest in [3, 17, 31, 54, 59] on how the solution of the interior impedance problem with piecewise-constant wavenumber (which is an approximation of the transmission problem) depends on the wavenumber, and the fact that the results in this section partially answer questions/conjectures from [3, 59].

6.1 Adapting the results of Popov and Vodev [57]

The paper [57] considers the Helmholtz transmission problem (2.2) with Ω_i a C^∞ convex domain with strictly positive curvature, $a_i = a_o = n_o = A_D = 1$ and $g_D = g_N = 0$; i.e., the BVP (3.7).

From our point of view, the importance of [57] is that their results can be adapted to show that if Ω_i is a C^∞ convex domain with strictly positive curvature, $n_i > 1$ and $A_N > 0$, then there exists an increasing sequence of real wavenumbers through which the solution operator grows super-algebraically; this is stated as Corollary 6.7 below. To state the results of [57] we need to define a *quasimode*.

Definition 6.1. (Quasimode for the BVP (3.7).) *A quasimode is a sequence*

$$\{k_j, (u_i^{(j)}, u_o^{(j)})\}_{j=1}^\infty$$

where $k_j \in \mathbb{C}$, $|k_j| \rightarrow \infty$, $\Re k_j \geq 1$, $u_{i/o}^{(j)} \in C^\infty(\overline{\Omega_{i/o}})$, the support of $u_{i/o}^{(j)}$ is contained in a fixed compact neighbourhood of Γ , $\|u_i^{(j)}\|_{L^2(\Gamma)} = 1$,

$$\left\| (\Delta + n_i k_j^2) u_i^{(j)} \right\|_{L^2(\Omega_i)} = \mathcal{O}(|k_j|^{-\infty}), \quad (6.1a)$$

$$\left\| (\Delta + k_j^2) u_o^{(j)} \right\|_{L^2(\Omega_o)} = \mathcal{O}(|k_j|^{-\infty}), \quad (6.1b)$$

$$\left\| u_i^{(j)} - u_o^{(j)} \right\|_{H^2(\Gamma)} = \mathcal{O}(|k_j|^{-\infty}), \quad \text{and} \quad (6.1c)$$

$$\left\| \partial_{\mathbf{n}} u_i^{(j)} - A_N \partial_{\mathbf{n}} u_o^{(j)} \right\|_{H^2(\Gamma)} = \mathcal{O}(|k_j|^{-\infty}), \quad (6.1d)$$

where, given an infinite sequence of complex numbers $\{z_j\}_{j=1}^\infty$, $z_j = \mathcal{O}(|k_j|^{-\infty})$ if for every $N > 0$ there exists a $C_N > 0$ such that $|z_j| \leq C_N |k_j|^{-N}$.

The significance of the assumption $\Re(k_j) \geq 1$ in Definition 6.1 is that it (a) bounds the wavenumbers away from zero, and (b) specifies that we are considering wavenumbers in the right-half complex plane. Therefore $\Re(k_j) \geq 1$ could be replaced by $\Re(k_j) \geq k_0$ for any $k_0 > 0$.

The concentration of the quasimodes near the boundary Γ means that they are understood in the asymptotic literature as “whispering gallery” modes; see, e.g., [2].

Theorem 6.2. (Existence of quasimodes [57].) *If Ω_i is a C^∞ convex domain with strictly positive curvature, $n_i > 1$ and $A_N > 0$, then there exists a quasimode for the transmission problem (3.7). Furthermore, $0 > \Im k_j = \mathcal{O}(|k_j|^{-\infty})$ (i.e. k_j is super-algebraically close to the real axis).*

The main result in [57] about resonances then follows from showing that there exists an infinite sequence of resonances that are super-algebraically close to the quasimodes [57, Proposition 2.1].

Theorem 6.2 implies that there exists an increasing sequence of *complex* wavenumbers through which the solution operator grows super-algebraically. We show in Corollary 6.7 below that this result implies that there exists an increasing sequence of *real* wavenumbers through which one obtains this growth.

To prove this corollary we need two preparatory results. The first (Corollary 6.3 below) is that one can change the normalisation $\|u_i^{(j)}\|_{L^2(\Gamma)} = 1$ in the definition of the quasimode to $\|\nabla_T u_i^{(j)}\|_{L^2(\Gamma)} + |k_j| \|u_i^{(j)}\|_{L^2(\Gamma)} = |k_j|$; it turns out that it will be more convenient for us to work with this normalisation (note that we put the factor $|k_j|$ on the right-hand side because, since we expect $\|\nabla_T u_i^{(j)}\|_{L^2(\Gamma)}$ to be proportional to $|k_j| \|u_i^{(j)}\|_{L^2(\Gamma)}$, this normalisation therefore keeps $\|u_i^{(j)}\|_{L^2(\Gamma)}$ being $\mathcal{O}(1)$). The second result (Lemma 6.4) is that, under the new normalisation, the L^2 norms of $u_{i/o}^{(j)}$ in $\Omega_{i/o}$ are bounded by $C|k_j|^m$ for some C and m independent of j and k_j .

Corollary 6.3. (Quasimode under different normalisation.) *If the normalisation $\|u_i^{(j)}\|_{L^2(\Gamma)} = 1$ in Definition 6.1 is changed to*

$$\|\nabla_T u_i^{(j)}\|_{L^2(\Gamma)} + |k_j| \|u_i^{(j)}\|_{L^2(\Gamma)} = |k_j|, \quad (6.2)$$

then the result of Theorem 6.2 still holds.

This follows from the construction of the quasimode in [57, §5.4]; instead of dividing the f_j s by $\|f_j\|_{L^2(\Gamma)}$ one divides by $(\|\nabla_T f_j\|_{L^2(\Gamma)} + |k_j| \|f_j\|_{L^2(\Gamma)})/|k_j|$. Since $\|f_j\|_{L^2(\Gamma)}$ is not super-algebraically small by [57, last equation on page 437], neither is $(\|\nabla_T f_j\|_{L^2(\Gamma)} + |k_j| \|f_j\|_{L^2(\Gamma)})/|k_j|$.

Lemma 6.4. (Bound on the L^2 norms of $u_{i/o}^{(j)}$ in $\Omega_{i/o}$.) *The quasimode of Theorem 6.2 (under the normalisation (6.2) in Corollary 6.3) satisfies*

$$\|u_i^{(j)}\|_{L^2(\Omega_i)} + \|u_o^{(j)}\|_{L^2(\Omega_o)} \leq C|k_j|^{1/2} + \mathcal{O}(|k_j|^{-\infty}), \quad (6.3)$$

where C is independent of j and k_j .

Proof. The plan is to obtain k_j -explicit bounds on the Cauchy data of $u_i^{(j)}$ and $u_o^{(j)}$ and then use Green's integral representation and k_j -explicit bounds on layer potentials. In the proof we use the notation that $a \lesssim b$ if $a \leq Cb$ for some C independent of j and k_j (but not necessarily independent of n_i and A_N).

By the transmission condition (6.1c) and the normalisation (6.2), we have

$$\begin{aligned} \|\nabla_T u_o^{(j)}\|_{L^2(\Gamma)} + |k_j| \|u_o^{(j)}\|_{L^2(\Gamma)} &\leq \|\nabla_T u_i^{(j)}\|_{L^2(\Gamma)} + |k_j| \|u_i^{(j)}\|_{L^2(\Gamma)} + \mathcal{O}(|k_j|^{-\infty}) \\ &= |k_j| + \mathcal{O}(|k_j|^{-\infty}). \end{aligned}$$

The bound in [51, Lemma 5] on the Dirichlet-to-Neumann map for Ω_i that are star-shaped with respect to a ball (and thus, in particular, smooth convex Ω_i) implies that

$$\begin{aligned} \|\partial_{\mathbf{n}} u_o^{(j)}\|_{L^2(\Gamma)} &\lesssim \|\nabla_T u_o^{(j)}\|_{L^2(\Gamma)} + |k_j| \|u_o^{(j)}\|_{L^2(\Gamma)} + \mathcal{O}(|k_j|^{-\infty}) \\ &\lesssim |k_j| (1 + \mathcal{O}(|k_j|^{-\infty})) \end{aligned} \quad (6.4)$$

where the $\mathcal{O}(|k_j|^{-\infty})$ in the first line is the contribution from $(\Delta + k_j^2)u_o^{(j)}$. Note that [51, Lemma 5] is valid for real wavenumber, but an analogous bound holds for complex wavenumbers with sufficiently small $\mathcal{O}(1)$ imaginary parts – see [50, Theorem I.2D]. Via the transmission condition (6.1d), the bound (6.4) holds with $\partial_{\mathbf{n}} u_o^{(j)}$ replaced by $\partial_{\mathbf{n}} u_i^{(j)}$.

The result (6.3) then follows from (i) Green's integral representation (applied in both Ω_o and Ω_i), (ii) the classical bound on the free resolvent

$$\|\chi R(k) \chi\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \lesssim \frac{1}{|k|} \exp(a(\Im k)_-), \quad (6.5)$$

for some $a > 0$ (depending on χ), where $x_- = 0$ for $x \geq 0$ and $x_- = -x$ for $x < 0$, and χ is any cutoff function, and (iii) the bounds on the single- and double-layer potentials

$$\|\chi \mathcal{S}_k\|_{L^2(\Gamma) \rightarrow L^2(\mathbb{R}^d)} \lesssim |k|^{-1/2} \exp(a(\Im k)_-) \text{ and } \|\chi \mathcal{D}_k\|_{L^2(\Gamma) \rightarrow L^2(\mathbb{R}^d)} \lesssim |k|^{1/2} \exp(a(\Im k)_-). \quad (6.6)$$

For a proof of (6.5) see, e.g., [67, Theorems 3 and 4] or [24, Theorem 3.1]. The bounds (6.6) with $\Im k = 0$ are obtained from (6.5) in [61, Lemma 4.3]; the same proof goes through with $\Im k \neq 0$. \square

Remark 6.5. (Sharp bounds on the single- and double-layer operators.) The bounds (6.6) are not sharp in their k -dependence. Sharp bounds for real k are given in [34, Theorems 1.1, 1.3, and 1.4], and we expect analogous bounds to be valid in the case when k is complex with sufficiently-small imaginary part [64]. In this case, the exponent $1/2$ in (6.3) would be lowered to $1/6$, but this would not affect the following result (Corollary 6.6), which uses (6.3).

Corollary 6.6. (Existence of quasimodes with real wavenumbers.) Given the quasimode in Theorem 6.2 (under the normalisation (6.2)),

$$\{\Re k_j, (u_i^{(j)}, u_o^{(j)})\}_{j=1}^\infty$$

is also a quasimode (again under the normalisation in Corollary 6.3).

Proof. From Definition 6.1, we only need to show that $\|(\Delta + n_i(\Re k_j)^2)u_i^{(j)}\|_{L^2(\Omega_i)} = \mathcal{O}(|k_j|^{-\infty})$ and $\|(\Delta + (\Re k_j)^2)u_o^{(j)}\|_{L^2(\Omega_o)} = \mathcal{O}(|k_j|^{-\infty})$. We have

$$\|(\Delta + n_i(\Re k_j)^2)u_i^{(j)}\|_{L^2(\Omega_i)} \leq \|n_i(-2i(\Re k_j)(\Im k_j) + (\Im k_j)^2)u_i^{(j)}\|_{L^2(\Omega_i)} + \mathcal{O}(|k_j|^{-\infty}),$$

which is $\mathcal{O}(|k_j|^{-\infty})$ since $\Im k_j = \mathcal{O}(|k_j|^{-\infty})$ and the L^2 norm of $u_i^{(j)}$ is bounded by $C|k_j|^{1/2}$ by the bound (6.3). Similarly, we have $\|(\Delta + (\Re k_j)^2)u_o^{(j)}\|_{L^2(\Omega_o)} = \mathcal{O}(|k_j|^{-\infty})$. \square

The super-algebraic growth of the solution operator through real values of k can therefore be summarised by the following.

Corollary 6.7. (Super-algebraic growth through real wavenumbers.) If Ω_i is a C^∞ convex domain with strictly positive curvature, $n_i > 1$ and $A_N > 0$, then there exists a sequence

$$\{\tilde{k}_j, (u_i^{(j)}, u_o^{(j)})\}_{j=1}^\infty,$$

where $\tilde{k}_j \in \mathbb{R}$, $\tilde{k}_j \rightarrow \infty$, $u_{i/o}^{(j)} \in C^\infty(\overline{\Omega_{i/o}})$, the support of $u_{i/o}^{(j)}$ is contained in a fixed compact neighbourhood of Γ , and for all $N \in \mathbb{N}$ there exists $C_N > 0$ independent of \tilde{k}_j such that

$$\begin{aligned} & \min \left\{ \|u_i^{(j)}\|_{H^{3/2}(\Omega_i)}, \|u_o^{(j)}\|_{H^{3/2}(\Omega_o)} \right\} \\ & \geq C_N \tilde{k}_j^N \left(\|(\Delta + n_i \tilde{k}_j^2)u_i^{(j)}\|_{L^2(\Omega_i)} + \|(\Delta + \tilde{k}_j^2)u_o^{(j)}\|_{L^2(\Omega_o)} \right. \\ & \quad \left. + \|u_i^{(j)} - u_o^{(j)}\|_{H^2(\Gamma)} + \|\partial_{\mathbf{n}} u_i^{(j)} - A_N \partial_{\mathbf{n}} u_o^{(j)}\|_{H^2(\Gamma)} \right). \end{aligned}$$

Proof. This follows from Corollary 6.6 by the trace theorem (see, e.g., [44, Theorem 3.37]). \square

6.2 The results of [8, 9] when Ω_i is a ball and numerical examples

The results of [8, 9] (summarised in [1, Chapter 5]) consider the case when Ω_i is a 2- or 3-d ball. It is convenient to discuss these alongside numerical examples in 2-d. (We therefore only discuss the 2-d results of [8], but the 3-d analogues of these are in [9].)

When Ω_i is the 2-d unit ball ($\Omega_i = B_1$), u_i and u_o can be expressed in terms of Fourier series $e^{i\nu\theta}$, where $\nu \in \mathbb{Z}$ and θ is the angular polar coordinate, with coefficients given in terms of Bessel and Hankel functions; see, e.g., [28, §12] or [1, §5.3]. The resonances of the BVP (3.7) (with $n_i > 1$) are then the (complex) zeros of

$$F_\nu(k) := A_N \sqrt{n_i} J'_\nu(\sqrt{n_i} k) H_\nu^{(1)}(k) - H_\nu^{(1)'}(k) J_\nu(\sqrt{n_i} k), \quad \nu \in \mathbb{Z},$$

see, e.g., [28, Equation (81)]. The asymptotics of the zeros of $F_\nu(k)$ in terms of ν are given by, e.g., [42, 60]. Indeed, let $k_{\nu,m}$ denote the m th zero of $F_\nu(k)$ with positive real part (where the zeros are ordered in terms of magnitude), and let α_m denote the m th zero of the Airy function $\text{Ai}(-z)$. Then by [42, Equation (1.1)], for fixed m ,

$$\sqrt{n_i} k_{\nu,m} = \nu + 2^{-1/3} \alpha_m \nu^{1/3} + \mathcal{O}(1) \quad \text{as } \nu \rightarrow \infty; \quad (6.7)$$

recall that the result [57, Proposition 2.1] (discussed in §6.1) implies that the resonances are real to all algebraic orders.

Exponential growth in k . For our first numerical examples, we take $n_i > 1$, $n_o = 1$, $A_N = 1$ and for $\nu \in \mathbb{N}_0$ we let k be the real part of the first resonance corresponding to the angular dependence $e^{i\nu\theta}$ (i.e. $k = \Re k_{\nu,1}$ in the previous paragraph). We then let $f_o = 0$ and $f_i = c_\nu J_\nu(kr)e^{i\nu\theta}$, where $c_\nu = (\pi(J_\nu^2(k) - J_{\nu+1}(k)J_{\nu-1}(k)))^{-1/2}$ is such that $\|f_i\|_{L^2(\Omega_i)} = 1$ for all $\nu \in \mathbb{N}_0$. The field u is then

$$u_i = c_\nu \frac{e^{i\nu\theta}}{k^2(n_i-1)} \left(J_\nu(kr) - J_\nu(k\sqrt{n_i}r) \frac{A_N J'_\nu(k) H_\nu^{(1)}(k) - H_\nu^{(1)'}(k) J_\nu(k)}{F_\nu(k)} \right),$$

$$u_o = c_\nu \frac{e^{i\nu\theta} H_\nu^{(1)}(kr)}{k^2(n_i-1) H_\nu^{(1)}(k)} \left(J_\nu(k) - J_\nu(k\sqrt{n_i}) \frac{A_N J'_\nu(k) H_\nu^{(1)}(k) - H_\nu^{(1)'}(k) J_\nu(k)}{F_\nu(k)} \right).$$

Figure 1 plots the $k\sqrt{n}$ -weighted L^2 norm and the H^1 seminorm of the particular $u_{i/o}$ above in Ω_i and D_R respectively (with R taken to be 2) as ν runs from 0 to ν_{\max} . The left and right panels in Figure 1 show the norms of $u_{i/o}$ for $n_i = 3$ and $\nu_{\max} = 64$, and for $n_i = 10$ and $\nu_{\max} = 23$, respectively. We see that the norms of the solution appear to grow exponentially; the existence of a sequence growing super-algebraically is expected by Corollary 6.7, although we are considering lower-order norms than in this result. We note that the estimate (6.7) is not enough to deduce exponential growth of the scattered field for $k = \Re k_{\nu,1}$ and increasing ν , but a more refined analysis is needed (taking into account the fact that the imaginary parts of the resonances are superalgebraically small). The result [1, Theorem 5.4] proves that, at least in the case of plane-wave incidence, Sobolev norms of u_o on spherical surfaces in D_R sufficiently close to Γ grow exponentially through a sequence of real wavenumbers, where the wavenumbers are defined in terms of the resonances by [1, Lemma 5.16]; this result can then be used to prove exponential growth in the L^2 norm of u_o .

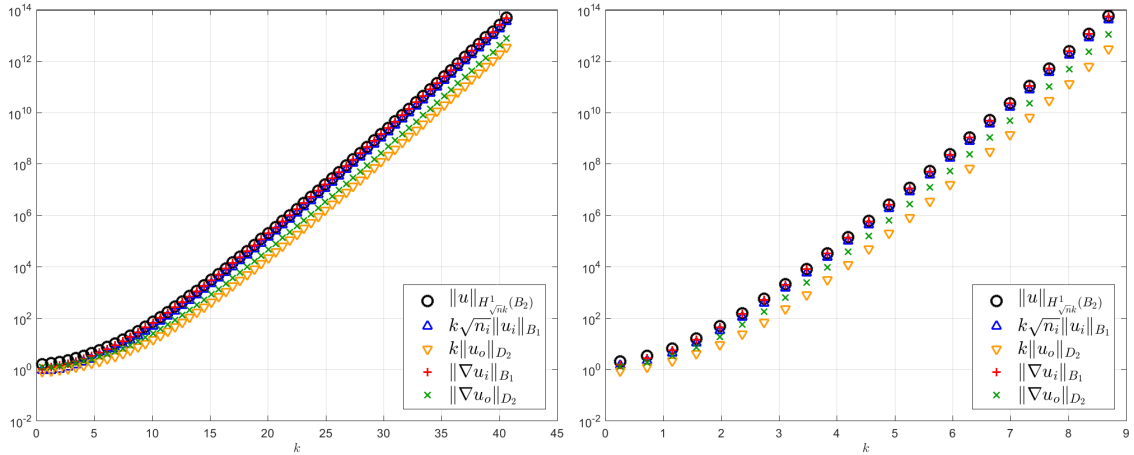


Figure 1: Left plot: the norms of the solution of BVP (3.7) with $\Omega_i = B_1$, $n_i = 3$, $A_N = 1$, $\|f_i\|_{\Omega_i} = 1$, $f_o = 0$, and where each k (on the abscissa) is the real part of the first resonance whose angular dependence is $e^{i\nu\theta}$ for $\nu = 0, \dots, 64$, i.e. $k = \Re k_{\nu,1}$. The black circles show the weighted $H^1(B_2)$ norms of u as in the left-hand side of (3.2), while the other four markers represent the weighted L^2 norms and the H^1 seminorms of u_i and u_o in $\Omega_i = B_1$ and $D_2 = B_2 \setminus \overline{B_1}$. Right plot: same as above with $n_i = 10$ and $\nu = 0, \dots, 23$.

Localisation of u_o in Ω_o at resonant frequencies. The plots in Figure 2 show the absolute value of the fields scattered by a plane wave impinging on the unit disc with $n_i = 100$. In the left plots, a wavenumber equal to the real part of a resonance $k_{\nu,m}$ excites a quasimode. In both these examples, u_o is localised close to Ω_i ; this is expected both from Theorem 6.2 above, and from [1, Theorem 5.2]. Indeed, this latter result gives bounds on u_o for all values of k and n_i , but if $n_i > 1$ they hold only in the “far field”, i.e. at distance at least $(n_i - 1) \text{diam } \Omega_i / 2$ from Ω_i , showing that the quasimodes generate large fields in a small neighbourhood of Ω_i only.

Sensitivity to the wavenumber. The right plots in Figure 2 show how a small perturbation of the wavenumber k (e.g. by a relative factor of about $2.4 \cdot 10^{-12}$ from that in the upper left plot to

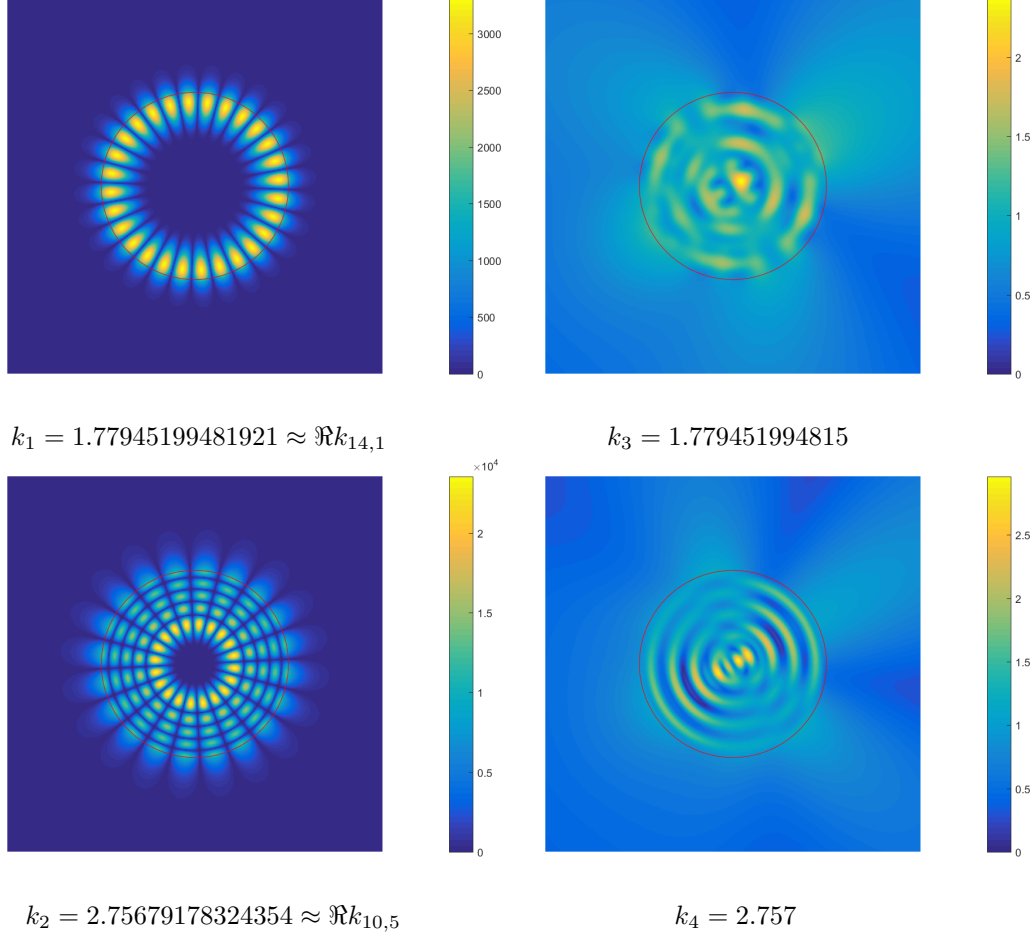


Figure 2: The absolute value of the field scattered by plane waves $e^{ik(x \cos \phi, y \sin \phi)}$ with unit amplitude and propagation direction $\phi = \pi/6$ hitting the disc B_1 with $n_i = 100$. The red line denotes the boundary Γ of $\Omega_i = B_1$. The four figures differ only in the wavenumber.

In the upper left plot, for $k_1 \approx \Re k_{14,1}$ the impinging plane wave excites a typical whispering gallery mode, quickly decaying away from the interface Γ . In the lower left plot, $k_2 \approx \Re k_{10,5}$, the absolute value of the excited mode has $m = 5$ peaks in the radial direction and $2\nu = 20$ in the angular direction. In the upper right plot k_3 differs from k_1 only by a factor of order 10^{-12} but nevertheless generates a completely different plot, as the quasimode is not excited; note also the different scales shown in the colour bar. In the lower right plot, k_4 differs from k_2 only by a factor of order 10^{-4} , and the same considerations apply.

that in the upper right one, and of $7.6 \cdot 10^{-5}$ in the second row) avoids the quasimode and gives a solution u with much smaller norm; see the scale displayed by the colour bar.

This phenomenon suggests that for certain data the exponential blow-up of the solution operator when $n_i > 1$ can be avoided. Indeed, given k , if $f_{i/o}(r, \theta) = \sum_{\ell} \alpha_{\nu, i/o}(r) e^{i\nu\theta}$ and $\alpha_{\nu, i/o} \equiv 0$ for any ν such that there exists an m such that $k_{\nu, m} \approx k$, then the resonant modes should be excluded from the solution. In the context of scattering by an incident field, [8, Theorem 6.5] describes the size of the neighbourhood I of $\{\Re k_{\nu, m}\}_{\nu \in \mathbb{Z}, m \in \mathbb{N}}$, such that the scattered field is uniformly bounded for $k \in (0, \infty) \setminus I$ (see also [1, Remark 5.5]). The strong sensitivity of the quasimodes with respect to the wavenumber explains why their presence can go unnoticed even by extensive numerical calculation; see, e.g., [3] for a related class of problems.

A Proof of Lemma 2.7 (existence, uniqueness, and regularity of the BVP solution)

Uniqueness: In this setting of Lipschitz Ω_i , uniqueness follows from Green's identity and a classical result of Rellich (given in, e.g., [21, Lemma 3.11 and Theorem 3.12]); the proof for our

assumptions on the parameters $k, n_i, n_o, a_o, a_i, A_D$, and A_N is short, and so we give it here.

With $f_i = 0, f_o = 0, g_D = 0, g_N = 0$, we apply Green's identity to u_i in Ω_i (this is allowed since Ω_i is Lipschitz and $u_i \in H^1(\Omega_i, \Delta)$ by, e.g., [44, Theorem 4.4]) to obtain

$$\int_{\Gamma} a_i u_i \overline{\partial_{\mathbf{n}} u_i} - a_i \int_{\Omega_i} |\nabla u|^2 + n_i \overline{k^2} \int_{\Omega_i} |u|^2 = 0.$$

Using the transmission conditions in (2.2) we get

$$\frac{a_o}{A_D A_N} \int_{\Gamma} u_o \overline{\partial_{\mathbf{n}} u_o} - a_i \int_{\Omega_i} |\nabla u|^2 + n_i \overline{k^2} \int_{\Omega_i} |u|^2 = 0. \quad (\text{A.1})$$

By [21, Theorem 3.12], if

$$\Im \left(k \sqrt{\frac{n_o}{a_o}} \int_{\Gamma} u_o \overline{\partial_{\mathbf{n}} u_o} \right) \geq 0$$

then $u_o \equiv 0$ in Ω_o ; the Cauchy data of u_i is then zero by the transmission conditions, and thus $u_i \equiv 0$ in Ω_i . Note that [21, Theorem 3.12] is stated and proved for $d = 3$, but the proof goes through in an identical way for $d = 2$ and $d \geq 4$ using the radiation condition (2.1) and the asymptotics of the fundamental solution in these dimensions (see, e.g., [24, Theorem 3.5]).

Multiplying (A.1) by $k \sqrt{n_o/a_o} (A_N A_D/a_o)$ and recalling that all the parameters apart from k are real, we see that a sufficient condition for uniqueness is

$$\Im \left(k a_i \int_{\Omega_i} |\nabla u|^2 - k n_i \overline{k^2} \int_{\Omega_i} |u|^2 \right) \geq 0;$$

this inequality holds since n_i and a_i are real, and $\Im k \geq 0$.

Existence and regularity: At least in the case $g_D = g_N = 0$, existence immediately follows from uniqueness via Fredholm theory. Here, however, we use the integral-equation argument of [66], since this also establishes the regularity results on Γ . This integral-equation argument is the Lipschitz analogue of the argument for sufficiently smooth Γ in [41, Theorem 4.6] (see also [23, Corollary 4.6], which covers 2-d polygons).

If $f_i = 0$ and $f_o = 0$ then the existence and the regularity of u follow from the integral-equation argument in [66, Theorem 7.2]; to match their notation we choose $u_e = \frac{a_o}{A_N a_i} u_o$ and

$$k_1 = k \sqrt{\frac{n_i}{a_i}}, \quad k_2 = k \sqrt{\frac{n_o}{a_o}}, \quad \mu_1 = A_D a_o, \quad \mu_2 = A_N a_i, \quad f = a_o g_D, \quad g = \frac{1}{A_N a_i} g_N.$$

The result [66, Theorem 7.2] is stated only for $d = 3$ and $k \in \mathbb{R} \setminus \{0\}$, but we now outline why it also holds for $d \geq 2$ and $\Im k > 0$; we first discuss the dimension.

The two reasons Torres and Welland only consider $d = 3$ is that

1. Their argument treats the Helmholtz integral operators as perturbations of the corresponding Laplace ones, and there is a technical difficulty that the fundamental solution of Laplace's equation does not tend to zero at infinity when $d = 2$, whereas it does for $d \geq 3$.
2. The case $d \geq 4$ is very similar to $d = 3$, except that the bounds on the kernels of the integral operators are slightly more involved.

Regarding the second reason: the only places in [66] where these bounds are used and that contribute to the existence result [66, Theorem 7.2] are Part (vi) of Lemma 6.2, and Points (i)–(vi) of Page 1466. The analogue of the Hankel-function bounds in [14, Equation 2.25] for $d \geq 4$ can be used to show that these results in [66] hold for $d \geq 4$.

Regarding the first reason: the harmonic-analysis results about Laplace integral operators, upon which Torres and Welland's proof rests, also hold when $d = 2$. More specifically, the results in Sections 4 and 6 of [66] hold when $d = 2$ by [68] and [20] (a convenient summary of these harmonic-analysis results is given in [14, Chapter 2]). The results (i) and (iii)–(vi) on Page 1463 of [66] hold when $d = 2$ by results in [68, §4], and the result (ii) holds if one defines the Laplace fundamental solution as $(1/2\pi) \log(a/|\mathbf{x} - \mathbf{y}|)$ where the constant a is not equal to the so-called ‘‘capacity’’ of Γ ; see [14, Page 115], [44, Theorem 8.16]. Finally, Lemma 3.1 of [66] holds when $d = 2$ by [25, Theorem 2]. All this means that Lemma 3.2 of [66] holds when $d = 2$, which in turn means that [66, Theorem 7.2] holds when $d = 2$.

We therefore have that [66, Theorem 7.2] holds for $d \geq 2$ and for $k \in \mathbb{R} \setminus \{0\}$. Inspecting the proof of [66, Theorem 7.2] we see that the properties of the Helmholtz boundary-integral operators used are unchanged if $\Im k > 0$. Indeed, the compactness of the differences of the Helmholtz and Laplace boundary-integral operators holds for $\Im k > 0$ by (v) and (vi) on Page 1466 of [66] and by (vi) in Lemma 6.2 of [66], and the uniqueness of the BVP (used on Page 1483) holds by the argument above. Therefore [66, Theorem 7.2] holds for $d \geq 2$ and for $k \in \mathbb{C} \setminus \{0\}$ with $\Im k \geq 0$.

If the volume source terms f_i and f_o are different from zero, we define w_i and w_o to be the solutions to the following problems:

$$\begin{cases} w_i \in H^1(\Omega_i) \\ a_i \Delta w_i + k^2 n_i w_i = f_i & \text{in } \Omega_i, \\ \partial_{\mathbf{n}} w_i - i k w_i = 0 & \text{on } \Gamma. \end{cases} \quad \begin{cases} w_o \in H_{\text{loc}}^1(\Omega_o) \\ a_o \Delta w_o + k^2 n_o w_o = f_o & \text{in } \Omega_o, \\ w_o = 0 & \text{on } \Gamma, \\ w_o \in \text{SRC}(k\sqrt{n_o/a_o}). \end{cases}$$

Then by [48, Proposition 3.2] $w_i \in H^1(\Omega_i)$, $\partial_{\mathbf{n}} w_i \in L^2(\Gamma)$ and $w_i \in H^1(\Gamma)$, and by [61, Part (i) of Lemma 3.5] $w_o \in H_{\text{loc}}^1(\Omega_o)$, $\partial_{\mathbf{n}} w_o \in L^2(\Gamma)$ and $w_o \in H^1(\Gamma)$ (note that both of the results [48, Proposition 3.2] and [61, Part (i) of Lemma 3.5] are specialisations of the regularity results of Nečas [52, §5.1.2 and §5.2.1] to Helmholtz BVPs). Then $\tilde{u}_i := u_i - w_i$ and $\tilde{u}_o := \frac{a_o}{A_N a_i}(u_o - w_o)$ satisfy problem (P) of [66] with k_1, k_2, μ_1, μ_2 as above and

$$f = a_o g_D + A_D a_o w_i \in H^1(\Gamma), \quad g = \frac{g_N - a_o \partial_{\mathbf{n}} w_o}{A_N a_i} + \partial_{\mathbf{n}} w_i \in L^2(\Gamma).$$

The existence and regularity of u_i, u_o follow by applying again Theorem 7.2 of [66] to \tilde{u}_i, \tilde{u}_o .

Acknowledgements

Over the last few years, several colleagues have asked the authors questions about wavenumber-explicit bounds on Helmholtz transmission problems; these include Leslie Greengard (New York University), Samuel Groth (MIT), David Hewett (University College London), and Ralf Hiptmair (ETH Zürich). We thank Giovanni S. Alberti (Genova), Simon Chandler-Wilde (University of Reading), Jeffrey Galkowski (McGill University), Ivan Graham (University of Bath), Zoïs Motier (Rennes), Melissa Tacy (Australian National University), and Georgi Vodev (Université de Nantes) for useful discussions. A. Moiola acknowledges support from EPSRC through the project EP/N019407/1 and from GNCS/INDAM.

References

- [1] G. S. ALBERTI AND Y. CAPDEBOSCQ, *Lectures on elliptic methods for hybrid inverse problems*, vol. 25 of Cours Spécialisés [Specialized Courses], Société Mathématique de France, Paris, 2018.
- [2] V. M. BABICH AND V. S. BULDYREV, *Short-Wavelength Diffraction Theory: Asymptotic Methods*, Springer-Verlag, Berlin, 1991.
- [3] H. BARUCQ, T. CHAUMONT-FRELET, AND C. GOUT, *Stability analysis of heterogeneous Helmholtz problems and finite element solution based on propagation media approximation*, Math. Comp., 86 (2017), pp. 2129–2157.
- [4] D. BASKIN, E. A. SPENCE, AND J. WUNSCH, *Sharp high-frequency estimates for the Helmholtz equation and applications to boundary integral equations*, SIAM Journal on Mathematical Analysis, 48 (2016), pp. 229–267.
- [5] M. BELLASSOUED, *Carleman estimates and distribution of resonances for the transparent obstacle and application to the stabilization*, Asymptotic Analysis, 35 (2003), pp. 257–279.
- [6] C. O. BLOOM AND N. D. KAZARINOFF, *A priori bounds for solutions of the Dirichlet problem for $[\Delta + \lambda^2 n(x)]u = f(x, \lambda)$ on an exterior domain*, Journal of Differential Equations, 24 (1977), pp. 437–465.
- [7] Y. BOUBENDIR, V. DOMINGUEZ, D. LEVADOUX, AND C. TURC, *Regularized combined field integral equations for acoustic transmission problems*, SIAM Journal on Applied Mathematics, 75 (2015), pp. 929–952.

- [8] Y. CAPDEBOSCQ, *On the scattered field generated by a ball inhomogeneity of constant index*, Asymptot. Anal., 77 (2012), pp. 197–246.
- [9] Y. CAPDEBOSCQ, G. LEADBETTER, AND A. PARKER, *On the scattered field generated by a ball inhomogeneity of constant index in dimension three*, in Multi-scale and high-contrast PDE: from modelling, to mathematical analysis, to inversion, vol. 577 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2012, pp. 61–80.
- [10] F. CARDOSO, G. POPOV, AND G. VODEV, *Distribution of resonances and local energy decay in the transmission problem II*, Mathematical Research Letters, 6 (1999), pp. 377–396.
- [11] F. CARDOSO, G. POPOV, AND G. VODEV, *Asymptotics of the number of resonances in the transmission problem*, Communications in Partial Differential Equations, 26 (2001), pp. 1811–1859.
- [12] F. CARDOSO AND G. VODEV, *Boundary stabilization of transmission problems*, Journal of Mathematical Physics, 51 (2010), p. 023512.
- [13] S. CHANDLER-WILDE AND B. ZHANG, *Electromagnetic scattering by an inhomogeneous conducting or dielectric layer on a perfectly conducting plate*, in Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, vol. 454, The Royal Society, 1998, pp. 519–542.
- [14] S. N. CHANDLER-WILDE, I. G. GRAHAM, S. LANGDON, AND E. A. SPENCE, *Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering*, Acta Numerica, 21 (2012), pp. 89–305.
- [15] S. N. CHANDLER-WILDE AND P. MONK, *Wave-number-explicit bounds in time-harmonic scattering*, SIAM Journal on Mathematical Analysis, 39 (2008), pp. 1428–1455.
- [16] S. N. CHANDLER-WILDE AND B. ZHANG, *Scattering of electromagnetic waves by rough interfaces and inhomogeneous layers*, SIAM Journal on Mathematical Analysis, 30 (1999), pp. 559–583.
- [17] T. CHAUMONT-FRELET, *On high order methods for the heterogeneous Helmholtz equation*, Computers & Mathematics with Applications, 72 (2016), pp. 2203–2225.
- [18] X. CLAEYS, R. HIPTMAIR, AND C. JEREZ-HANCKES, *Multi-trace boundary integral equations*, in Direct and Inverse Problems in Wave Propagation and Applications, I. Graham, U. Langer, J. Melenk, and M. Sini, eds., no. 14 in Radon Ser. Comput. Appl. Math., De Gruyter, Berlin, 2013.
- [19] X. CLAEYS, R. HIPTMAIR, C. JEREZ-HANCKES, AND S. PINTARELLI, *Novel multitrace boundary integral equations for transmission boundary value problems*, in Unified Transform for Boundary Value Problems: Applications and Advances, A. S. Fokas and B. Pelloni, eds., SIAM, 2015.
- [20] R. R. COIFMAN, A. MCINTOSH, AND Y. MEYER, *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Annals of Mathematics, 116 (1982), pp. 361–387.
- [21] D. L. COLTON AND R. KRESS, *Integral Equation Methods in Scattering Theory*, John Wiley & Sons Inc., New York, 1983.
- [22] M. COSTABEL AND M. DAUGE, *Un résultat de densité pour les équations de Maxwell régularisées dans un domaine lipschitzien*, C. R. Acad. Sci. Paris Sér. I Math., 327 (1998), pp. 849–854.
- [23] M. COSTABEL AND E. STEPHAN, *A direct boundary integral equation method for transmission problems*, Journal of Mathematical Analysis and Applications, 106 (1985), pp. 367–413.
- [24] S. DYATLOV AND M. ZWORSKI, *Mathematical theory of scattering resonances*, <http://math.mit.edu/~dyatlov/res/>, 2018. Book in progress.
- [25] L. ESCAURIAZA, E. FABES, AND G. VERCHOTA, *On a regularity theorem for weak solutions to transmission problems with internal Lipschitz boundaries*, Proceedings of the American Mathematical Society, 115 (1992), pp. 1069–1076.
- [26] E. FOUASSIER, *Morrey–Campanato estimates for Helmholtz equations with two unbounded media*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 135 (2005), pp. 767–776.
- [27] J. GALKOWSKI, *Personal communication*.

- [28] ———, *The Quantum Sabine Law for Resonances in Transmission Problems*, Pure and Applied Analysis, to appear (2018).
- [29] M. GANDER AND H. ZHANG, *Iterative Solvers for the Helmholtz Equation: Factorizations, Sweeping Preconditioners, Source Transfer, Single Layer Potentials, Polarized Traces, and Optimized Schwarz Methods*, SIAM Review, to appear (2018).
- [30] I. GRAHAM, O. PEMBERY, AND E. SPENCE, *The helmholtz equation in heterogeneous media: A priori bounds, well-posedness, and resonances*, Journal of Differential Equations, (2018).
- [31] I. G. GRAHAM AND S. SAUTER, *Stability and error analysis for the Helmholtz equation with variable coefficients*, arXiv preprint arXiv:1803.00966, (2018).
- [32] S. P. GROTH, D. P. HEWETT, AND S. LANGDON, *Hybrid numerical-asymptotic approximation for high-frequency scattering by penetrable convex polygons*, IMA Journal of Applied Mathematics, 80 (2015), pp. 324–353.
- [33] S. P. GROTH, D. P. HEWETT, AND S. LANGDON, *A hybrid numerical-asymptotic boundary element method for high frequency scattering by penetrable convex polygons*, Wave Motion, 78 (2018), pp. 32–53.
- [34] X. HAN AND M. TACY, *Sharp norm estimates of layer potentials and operators at high frequency*, Journal of Functional Analysis, 269 (2015), pp. 2890–2926. With an appendix by Jeffrey Galkowski.
- [35] D. J. HANSEN, C. POIGNARD, AND M. S. VOGELIUS, *Asymptotically precise norm estimates of scattering from a small circular inhomogeneity*, Applicable Analysis, 86 (2007), pp. 433–458.
- [36] R. HIPTMAIR, A. MOIOLA, AND I. PERUGIA, *Stability results for the time-harmonic Maxwell equations with impedance boundary conditions*, Mathematical Models & Methods in Applied Sciences, 21 (2011), pp. 2263–2287.
- [37] R. HIPTMAIR, L. SCARABOSIO, C. SCHILLINGS, AND C. SCHWAB, *Large deformation shape uncertainty quantification in acoustic scattering*, Advances in Computational Mathematics, 44 (2018), pp. 1475–1518.
- [38] G. HU, X. LIU, F. QU, AND B. ZHANG, *Variational approach to scattering by unbounded rough surfaces with Neumann and generalized impedance boundary conditions*, Communications in Mathematical Sciences, 13 (2015), pp. 511–537.
- [39] T. HUTTUNEN, P. MONK, AND J. P. KAIPIO, *Computational aspects of the ultra-weak variational formulation*, Journal of Computational Physics, 182 (2002), pp. 27–46.
- [40] S. KIM AND J. E. PASCIAK, *The computation of resonances in open systems using a perfectly matched layer*, Mathematics of Computation, 78 (2009), pp. 1375–1398.
- [41] R. KRESS AND G. ROACH, *Transmission problems for the Helmholtz equation*, Journal of Mathematical Physics, 19 (1978), pp. 1433–1437.
- [42] C. LAM, P. LEUNG, AND K. YOUNG, *Explicit asymptotic formulas for the positions, widths, and strengths of resonances in Mie scattering*, Journal of the Optical Society of America B, 9 (1992), pp. 1585–1592.
- [43] A. LECHLEITER AND S. RITTERBUSCH, *A variational method for wave scattering from penetrable rough layers*, IMA Journal of Applied Mathematics, 75 (2010), pp. 366–391.
- [44] W. MCLEAN, *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, Cambridge, 2000.
- [45] R. B. MELROSE AND J. SJÖSTRAND, *Singularities of boundary value problems. I*, Communications on Pure and Applied Mathematics, 31 (1978), pp. 593–617.
- [46] ———, *Singularities of boundary value problems. II*, Communications on Pure and Applied Mathematics, 35 (1982), pp. 129–168.
- [47] A. MOIOLA, *Trefftz-discontinuous Galerkin methods for time-harmonic wave problems*, PhD thesis, Seminar for applied mathematics, ETH Zürich, 2011. Available at <http://e-collection.library.ethz.ch/view/eth:4515>.
- [48] A. MOIOLA AND E. A. SPENCE, *Is the Helmholtz equation really sign-indefinite?*, SIAM Review, 56 (2014), pp. 274–312.
- [49] C. S. MORAWETZ, *The decay of solutions of the exterior initial-boundary value problem for the wave equation*, Communications on Pure and Applied Mathematics, 14 (1961), pp. 561–568.

- [50] ———, *Decay for solutions of the exterior problem for the wave equation*, Communications on Pure and Applied Mathematics, 28 (1975), pp. 229–264.
- [51] C. S. MORAWETZ AND D. LUDWIG, *An inequality for the reduced wave operator and the justification of geometrical optics*, Communications on Pure and Applied Mathematics, 21 (1968), pp. 187–203.
- [52] J. NEČAS, *Les méthodes directes en théorie des équations elliptiques*, Masson, 1967.
- [53] H.-M. NGUYEN AND M. S. VOGELIUS, *Full range scattering estimates and their application to cloaking*, Arch. Ration. Mech. Anal., 203 (2012), pp. 769–807.
- [54] M. OHLBERGER AND B. VERFÜRTH, *A new heterogeneous multiscale method for the Helmholtz equation with high contrast*, Multiscale Modeling & Simulation. A SIAM Interdisciplinary Journal, 16 (2018), pp. 385–411.
- [55] B. PERTHAME AND L. VEGA, *Morrey–Campanato estimates for Helmholtz equations*, Journal of Functional Analysis, 164 (1999), pp. 340–355.
- [56] G. POPOV AND G. VODEV, *Distribution of the resonances and local energy decay in the transmission problem*, Asymptotic Analysis, 19 (1999), pp. 253–265.
- [57] ———, *Resonances near the real axis for transparent obstacles*, Communications in Mathematical Physics, 207 (1999), pp. 411–438.
- [58] F. RELICH, *Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral*, Mathematische Zeitschrift, 46 (1940), pp. 635–636.
- [59] S. SAUTER AND C. TORRES, *Stability estimate for the Helmholtz equation with rapidly jumping coefficients*, Zeitschrift für Angewandte Mathematik und Physik, 69 (2018), p. 69:139.
- [60] S. SCHILLER, *Asymptotic expansion of morphological resonance frequencies in Mie scattering*, Applied Optics, 32 (1993), pp. 2181–2185.
- [61] E. A. SPENCE, *Wavenumber-explicit bounds in time-harmonic acoustic scattering*, SIAM Journal on Mathematical Analysis, 46 (2014), pp. 2987–3024.
- [62] E. A. SPENCE, S. N. CHANDLER-WILDE, I. G. GRAHAM, AND V. P. SMYSHLYAEV, *A new frequency-uniform coercive boundary integral equation for acoustic scattering*, Communications on Pure and Applied Mathematics, 64 (2011), pp. 1384–1415.
- [63] E. A. SPENCE, I. V. KAMOTSKI, AND V. P. SMYSHLYAEV, *Coercivity of combined boundary integral equations in high-frequency scattering*, Communications on Pure and Applied Mathematics, 68 (2015), pp. 1587–1639.
- [64] M. TACY, *Personal communication*, 2016.
- [65] M. THOMAS, *Analysis of rough surface scattering problems*, PhD thesis, University of Reading, 2006.
- [66] R. H. TORRES AND G. V. WELLAND, *The Helmholtz equation and transmission problems with Lipschitz interfaces*, Indiana University Mathematics Journal, 42 (1993), pp. 1457–1485.
- [67] B. R. VAINBERG, *On the short wave asymptotic behaviour of solutions of stationary problems and the asymptotic behaviour as $t \rightarrow \infty$ of solutions of non-stationary problems*, Russian Mathematical Surveys, 30 (1975), pp. 1–58.
- [68] G. VERCHOTA, *Layer potentials and regularity for the Dirichlet problem for Laplace’s equation in Lipschitz domains*, Journal of Functional Analysis, 59 (1984), pp. 572–611.
- [69] G. VODEV, *On the uniform decay of the local energy*, Serdica Mathematical Journal, 25 (1999), pp. 191–206.
- [70] B. ZHANG AND S. N. CHANDLER-WILDE, *Acoustic scattering by an inhomogeneous layer on a rigid plate*, SIAM Journal on Applied Mathematics, 58 (1998), pp. 1931–1950.
- [71] ———, *A uniqueness result for scattering by infinite rough surfaces*, SIAM Journal on Applied Mathematics, 58 (1998), pp. 1774–1790.